Sublinear-time Algorithms

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Outline

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Definition:

- Algorithms which run in o(n) time without preprossesing of the input are called Sublinear - time Algorithms.
- Note that such algorithms do not read the entire input but only an infinitesimal part of it!

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- We also assume that we have access to all elements in the list
  - All n list elements are stored in an array (but the array is not sorted and we do not impose any order for the array elements).
- We can easily see that it is impossible to do the search in o(n) time using a deterministic algorithm.
  - However, if we allow randomization, then we can complete the search in $O(\sqrt{n})$ expected time

Randomized algorithm

- Sample uniformly at random a set S of $\Theta(\sqrt{n})$ elements from the input.
- Scan all the elements in S and in $O(\sqrt{n})$ time we can find the max $p \in S$ and the min $q \in S$ such that $p \leq x \leq q$.
- Traverse the input list starting at p until we find either the sought key x or we find element q.

Lemma 1

The algorithm above completes the search in expected $O(\sqrt{n})$ time.

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Proof

The running time of the algorithm if equal to $O(\sqrt{n})$ plus the number of the input elements between p and q. Since S contains $\Theta(\sqrt{n})$ elements, the expected number of input elements between p and q is $O(n/|S|) = O(\sqrt{n})$. This implies that the expected running time of the algorithm is $O(\sqrt{n})$.

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Given two convex polygons A and B in $\mathbb{R}^2$, each with n vertices, determine if they intersect, and if so, then find a point in their intersection.

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- In fact, within the same time one can either find a point that is in the intersection of A and B, or find a line L that separates A from B.
- Can we obtain a better running time?

$O(\sqrt{n})$ algorithm

We assume that A and B are given by their doubly-linked lists of vertices such that each vertex has as its successor the next vertex of the polygon in the clockwise order.

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Algorithm

- Sample uniformly at random $\Theta(\sqrt{n})$ vertices from each A and B, and let C_A and C_B be the convex hulls of the sample point sets for the polygons A and B, respectively.
- In $O(\sqrt{n})$ time we can check if C_A and C_B intersects.

- If they don't,let
  - L: the bitangent separating line returned by the algorithm.
  - a,b: The points in L that belong to A and B, respectively.
  - a_1, a_2:the two vertices adjacent to a in A.
  - P_A: We define polygon P_A by walking from a to a1 and then continue walking along the boundary of A until we cross L again (expected size: $O(\sqrt{n})$).

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To be continued on the whiteboard...

Lemma 2

The problem of determining whether two convex n-gons intersect can be solved in $O(\sqrt{n})$ expected time, which is asymptotically optimal.

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Assume we have access to the degree distribution of the vertices of an undirected connected graph $G = (V,E)$, i.e., for any vertex $v \in V$ we can query for its degree.

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- It seems that approximating the average degree is equivalent to approximating the average of a set of n numbers with values between 1 and n - 1, which is not possible in sublinear time.
- But our problem is much easier because the degrees of the vertices we do not sample depends on the degrees of the vertices we do sample!

Proposition

Let d denote the average degree in $G = (V, E)$ and let d_S denote the random variable for the average degree of a set S of s vertices chosen uniformly at random from V . We will show that if we set $s \geq \beta\sqrt{n}/\epsilon^{O(1)}$ for an appropriate constant β, then $d_S \geq (\frac{1}{2} - \epsilon) * d$ with probability at least $1 - \frac{\epsilon}{64}$.

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- The probability that one of them has too small average degree is at most $\frac{8}{\epsilon} * \frac{\epsilon}{64} = \frac{1}{8}$.
- Hence, the output value will satisfy both inequalities with probability at least $3/4$.
- This gives us a $(2+\epsilon)$-approximation algorithm.

Lower bound

- Let H be the set of the $\sqrt{\epsilon n}$ vertices with highest degree in G and let $L = V \setminus H$ be the set of the remaining vertices.

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- Let d_H be the degree of a vertex with the smallest degree in H.
- We assume that all sampled vertices come from the set L.
- Let X_i, $1 \leq i \leq s$, be the random variable for the degree of the ith vertex from S.

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- From Hoeffding bounds it follows that:
$$\Pr[\sum_{i=1}^{s} x_i \leq (1-\epsilon)E[\sum_{i=1}^{s} X_i]] \leq e^{-\frac{E[\sum_{i=1}^{s} X_i]\epsilon^2}{d_H}}$$

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- So, $E[X_i] \geq (\frac{1}{2}-\epsilon) * d_H * |H|/n$ and by linearity of expectation: $E[\sum_{i=1}^{s} X_i] \geq s * (\frac{1}{2}-\epsilon) * d_H * |H|/n$

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- So, $E[X_i] \geq (\frac{1}{2} - \epsilon) * d_H * |H|/n$ and by linearity of expectation: $E[\sum_{i=1}^{s} X_i] \geq s * (\frac{1}{2} - \epsilon) * d_H * |H|/n$
- By choosing s appropriately we can have $d_S \geq (1-\epsilon) * d$ with high probability (depending on s).

Minimum spanning trees

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- We assume that the graph is given in adjacency list representation, i.e., for every vertex v there is a list of its at most D neighbors, which can be accessed from v.
- It is possible to select a vertex uniformly at random.

MST cost approximation algorithm

Main Idea

Express the cost of a minimum spanning tree as the number of connected components in certain auxiliary subgraphs of G.

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- It can be shown that: $\mathrm{MST} = \mathrm{n} - \mathrm{W} + \sum_{\mathrm{i}=1}^{\mathrm{W}-1} \mathrm{c}^{(\mathrm{i})}$

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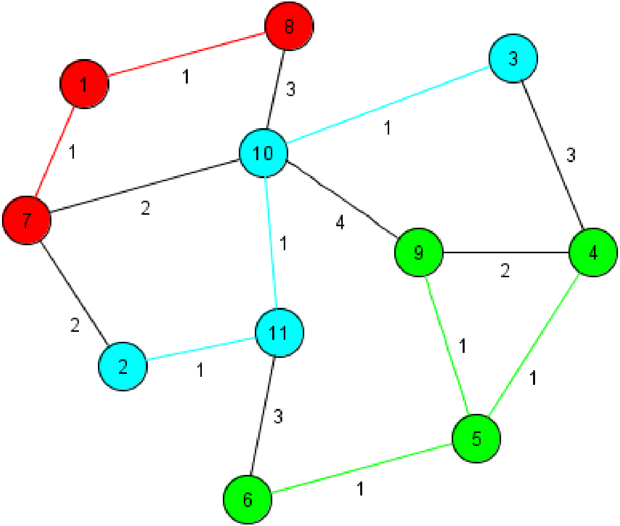
- It can be shown that: $\mathrm{MST} = \mathrm{n} - \mathrm{W} + \sum_{\mathrm{i}=1}^{\mathrm{W}-1} \mathrm{c}^{(\mathrm{i})}$
- So there is a simple algorithm for the approximation of MST weight.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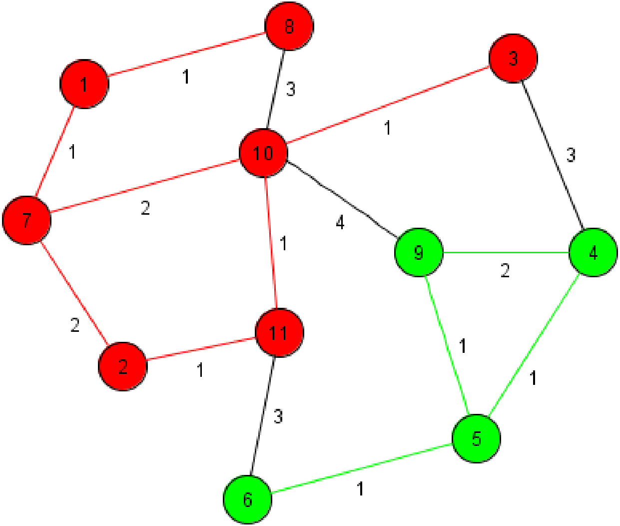
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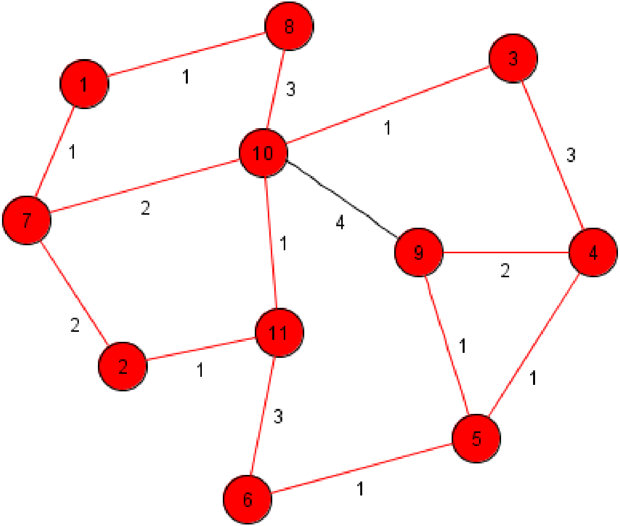
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Algorithm

APPROXMSTWEIGHT(G,ε) for i = 1 to W 1 Compute estimator $\underline{c^{(i)}}$ for $c^{(i)}$ output $\underline{\text{MST}} = n - W + \sum_{i=1}^{W-1} \underline{c^{(i)}}$

Approximation algorithm for connected components

Algorithm

APPROXCONNECTEDCOMPS(G, s) Input: an arbitrary undirected graph G
Output: $\underline{c}$: an estimation of the number of connected components of G
Choose s vertices $u_1, \ldots, u_s$ uniformly at random.
for i = 1 to s do choose X according to $\Pr[X \geq k] = 1/k$
run breadth-fist-search (BFS) starting at u_i until either (1) the whole connected component containing $u_i$has been explored, or (2) X vertices have been explored if BFS stopped in case (1) then bi = 1 else bi = 0
output $\underline{c} = \frac{n}{s}\sum_{i=1}^{s} b_i$

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- $E[b_i] = \sum_{connected components} Pr[u_i \in C] * Pr[X \geq |C|] =$
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- By linearity of expectation: $E[\underline{c}] = c$.

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- $\mathrm{E}[\mathrm{b_i}] = \sum_{\mathrm{connectedcomponents}} \mathrm{Pr}[\mathrm{u_i} \in \mathrm{C}] * \mathrm{Pr}[\mathrm{X} \geq |\mathrm{C}|] = \sum \frac{|\mathrm{C}|}{\mathrm{n}} * \frac{1}{|\mathrm{C}|} = \frac{\mathrm{c}}{\mathrm{n}}$
- By linearity of expectation: $\mathrm{E}[\underline{\mathrm{c}}] = \mathrm{c}$.
- $\mathrm{Var}[\mathrm{b_i}] = \mathrm{E}[\mathrm{b_i^2}] - \mathrm{E}[\mathrm{b_i}]^2 \leq \mathrm{E}[\mathrm{b_i^2}] = \mathrm{E}[\mathrm{b_i}] = \frac{\mathrm{c}}{\mathrm{n}}$

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$$\sum \frac{|C|}{n} * \frac{1}{|C|} = \frac{c}{n}$$
- By linearity of expectation: $E[\underline{c}] = c$.
- $Var[b_i] = E[b_i^2] - E[b_i]^2 \leq E[b_i^2] = E[b_i] = \frac{c}{n}$
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- $\mathrm{E}[\mathrm{b_i}] = \sum_{\mathrm{connectedcomponents}} \mathrm{Pr}[\mathrm{u_i} \in \mathrm{C}] * \mathrm{Pr}[\mathrm{X} \geq |\mathrm{C}|] =$
  $\sum \frac{|\mathrm{C}|}{\mathrm{n}} * \frac{1}{|\mathrm{C}|} = \frac{\mathrm{c}}{\mathrm{n}}$
- By linearity of expectation: $\mathrm{E}[\underline{\mathrm{c}}] = \mathrm{c}$.
- $\mathrm{Var}[\mathrm{b_i}] = \mathrm{E}[\mathrm{b_i^2}] - \mathrm{E}[\mathrm{b_i}]^2 \leq \mathrm{E}[\mathrm{b_i^2}] = \mathrm{E}[\mathrm{b_i}] = \frac{\mathrm{c}}{\mathrm{n}}$
- The $\mathrm{b_i}$ are mutually independent and so we have
- $\mathrm{Pr}[|\underline{\mathrm{c}} - \mathrm{E}[\underline{\mathrm{c}}]| \geq \lambda \mathrm{n}] \leq \frac{\mathrm{n*c}}{\mathrm{s}*\lambda^2*\mathrm{n}^2} \leq \frac{1}{\lambda^2 * \mathrm{s}}$

From this, it follows that one can approximate the number of connected components within additive error of $\lambda * \mathrm{n}$ in a graph with maximum degree D in $\mathrm{O}(\mathrm{s} * \mathrm{D} * \mathrm{logn}) = \mathrm{O}(\frac{\mathrm{D} * \mathrm{logn}}{\lambda^2 * \varrho})$ time and with probability $1 - \varrho$.

Metric Steiner tree

Steiner tree Problem definition

Given an undirected graph G=(V,E) with nonnegative edge costs and whose vertices are partitioned into two sets, R(equired) and S(teiner) find a minimum cost tree in G that contains all the required vertices and any subset of Steiner vertices.

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Metric Steiner tree

If the edge costs satisfy the triangle inequality $(\forall \mathrm{u}, \mathrm{v}, \mathrm{w} : \mathrm{cost}(\mathrm{u}, \mathrm{v}) \leq \mathrm{cost}(\mathrm{u}, \mathrm{w}) + \mathrm{cost}(\mathrm{w}, \mathrm{v}))$ we call that : Metric Steiner tree problem.

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There is an approximation factor preserving reduction from the Steiner tree to the metric Steiner tree problem.

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Construction

- Let G' be $K_{|V|}$
- Define: cost(u,v) (in G')= cost of the shortest path from u to v in G (G' is the metric closure of G).
- The sets R,S remain the same.
- Also, $OPT' \leq OPT$.

Proof

- Let T' be a Steiner tree in G'.
- Replace each edge of T' with the corresponding path of equal cost in T.
- Delete some edges to obtain a tree T.
- As we can see,$\forall \mathrm{T}' \exists \mathrm{T}$ such that $\mathrm{cost}(\mathrm{T}) \leq \mathrm{cost}(\mathrm{T}')$. So, $\mathrm{OPT} \leq \mathrm{OPT}'$.
- Finally OPT=OPT'. And this is an approximation factor preserving reduction.

MST-based algorithm

Theorem 3.3

The cost of an MST on R is within 2*OPT.

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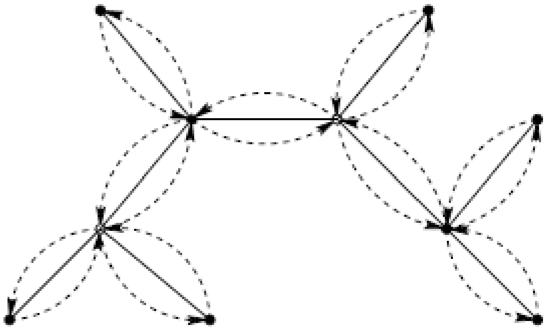
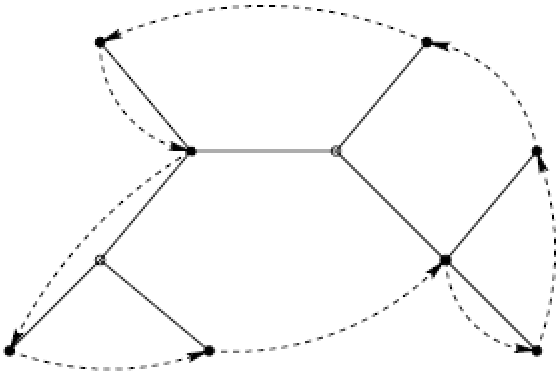
Proof

- Consider a Steiner tree of cost OPT. By doubling its edges we obtain an Eulerian graph connecting all vertices of R and, possibly, some Steiner vertices.
- Find an Euler tour of this graph.
- Next obtain a Hamiltonian cycle on the vertices of R by traversing the Euler tour and short-cutting Steiner vertices and previously visited vertices of R.

- Because of triangle inequality, the shortcuts do not increase the cost of the tour. If we delete one edge of this Hamiltonian cycle, we obtain a path that spans R and has cost at most 2 OPT. This path is also a spanning tree on R. Hence, the MST on R has cost at most 2 OPT.





Tight Example

For a tight example, consider a graph with n required vertices and one Steiner vertex. An edge between the Steiner vertex and a required vertex has cost 1, and an edge between two required vertices has cost 2 (not all edges of cost 2 are shown below). In this graph, any MST on R has cost $2(n-1)$, while $OPT = n$

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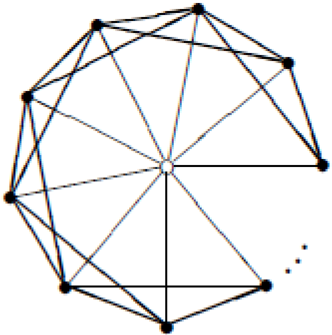

TSP

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Theorem 3.6

For any polynomial time computable function $\alpha(n)$, TSP cannot be approximated within a factor of $\alpha(n)$, unless $P = NP$.

Assume, for a contradiction, that there is a factor $\alpha(\mathrm{n})$ polynomial time approximation algorithm, A, for the general TSP problem. We will show that A can be used for deciding the Hamiltonian cycle problem (which is NP hard) in polynomial time, thus implying P = NP.

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Proof

- The central idea is a reduction from the Hamiltonian cycle problem to TSP, that transforms a graph G on n vertices to an edge-weighted complete graph G' on n vertices such that:
  - if G has a Hamiltonian cycle, then the cost of an optimal TSP tour in G' is n
  - if G does not have a Hamiltonian cycle, then an optimal TSP tour in G' is of cost $\geq \alpha(n) * n$.
- The reduction is simple. Assign a weight of 1 to edges of G, and a weight of $\alpha(n) * n$ to non-edges, to obtain G'.

A simple factor 2 algorithm

The lower bound we will use for obtaining this factor is the cost of an MST in G.

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Algorithm: Metric TSP factor 2

1. Find an MST, T, of G.
2. Double every edge of the MST to obtain an Eulerian graph.
3. Find an Eulerian tour, T , on this graph.
4. Output the tour that visits vertices of G in the order of their first appearance in T . Let C be this tour.

Theorem 3.8

Algorithm 3.7 is a factor 2 approximation algorithm for metric TSP.

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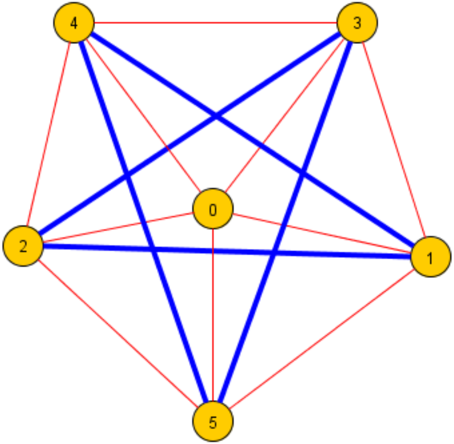
Proof

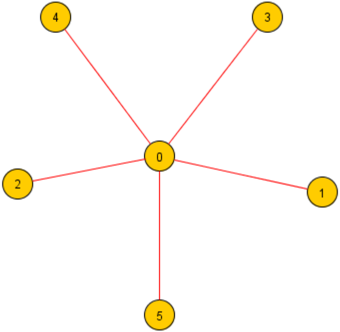
As noted above, cost(T) $\leq$ OPT. Since T' contains each edge of T twice, cost(T') = 2*cost(T). Because of triangle inequality, after the short-cutting step, cost(C) $\leq$ cost(T$'$). Combining these inequalities we get that cost(C) $\leq$ 2OPT.

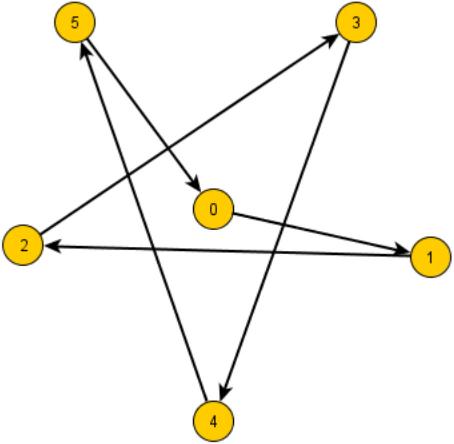
Tight Example

A tight example for this algorithm is given by a complete graph on n vertices with edges of cost 1 and 2. We present the graph for n = 6 below, where thick edges have cost 2 and remaining edges have cost 1. For arbitrary n the graph has 2n-2 edges of cost 1, with these edges forming the union of a star and an n 1 cycle; all remaining edges have cost 2.


4
3
0
2
1
5

Suppose that the MST found by the algorithm is the spanning star created by edges of cost 1. Moreover, suppose that the Euler tour constructed in Step 3 visits vertices in order shown next for n = 6:


0
1
2
3
4
5


5
3
0
2
1
4

Improving the factor to 3/2

Algorithm - factor 3/2

- Find an MST of G, say T.
- Compute a minimum cost perfect matching, M, on the set of odd-degree vertices of T. Add M to T and obtain an Eulerian graph.
- Find an Euler tour, T , of this graph.
- Output the tour that visits vertices of G in order of their first appearance in T . Let C be this tour.

Lemma 3.11

Let $V' \subseteq V$ be the set of odd-degree vertices of G ($|V'|$ is even) and let M be a minimum cost perfect matching on V'. Then, $\mathrm{cost}(M) \leq OPT/2$.

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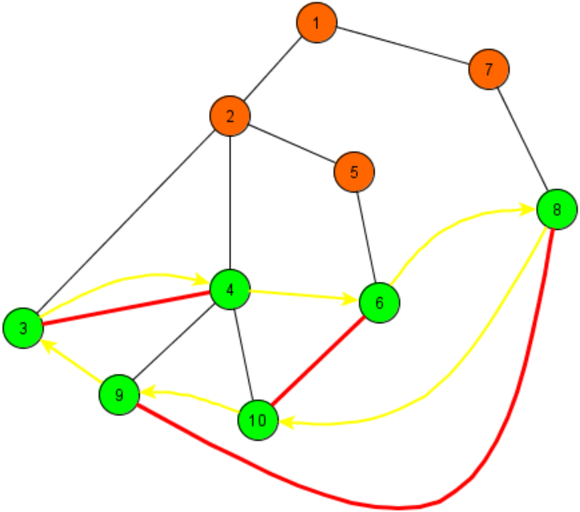
Proof

Let τ :optimal TSP tour. Let τ' be the tour on V' obtained by short-cutting τ. By the triangle inequality, $\text{cost}(\tau') \leq \text{cost}(\tau)$. Now, τ' is the union of two perfect matchings on V', each consisting of alternate edges of τ. Thus, the cheaper of these matchings has cost $\leq \text{cost}(\tau')/2 \leq \text{OPT}/2$


1
7
2
5
8
4
6
3
9
10

Theorem 3.12

Algorithm 3.10 achieves an approximation guarantee of 3/2 for metric TSP.

Theorem 3.12

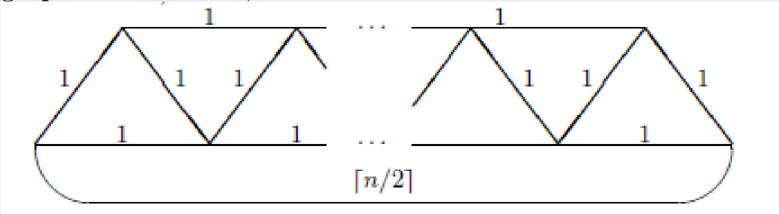
Algorithm 3.10 achieves an approximation guarantee of 3/2 for metric TSP.

Proof

The cost of the Euler tour,
$\mathrm{cost}(\mathrm{T}') \leq \mathrm{cost}(\mathrm{T}) + \mathrm{cost}(\mathrm{M}) \leq \mathrm{OPT} + \frac{1}{2}\mathrm{OPT} = \frac{3}{2}\mathrm{OPT}$, where the first inequality follows by using the two lower bounds on OPT. Using the triangle inequality, $\mathrm{cost}(\mathrm{C}) \leq \mathrm{cost}(\mathrm{T})$, and the theorem follows.

Tight Example

A tight example for this algorithm is given by the following graph on n vertices, with n odd:



Introduction

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- Surprisingly, these lower bounds do not necessarily hold when one wants to estimate the cost of an optimal solution.
- There is a constant factor approximation algorithm for the metric uncapacitated facility location problem with uniform costs and in which every point can open a facility, that runs in $O(n\log^2 n)$time, that is, in time sublinear in the input size.

Problem Definition

(Metric) Minimum Facility Location Problem

We are given a metric (P,D), and a subset $F \subseteq P$ of facilities. For each facility $v \in F$, we are given a non-negative cost f(v), and for each point $u \in P$, a nonnegative demand d(u). The problem consists of finding a set $F' \subseteq F$, so as to minimize I= $\sum_{v \in F'} f(v) + \sum_{u \in P} d(u) * D(u, F')$ where D(u, F') $= \min_{v \in F} D(u, v)$.

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- That is, $\forall v \in F$, $f(v) = c$ for some $c \geq 0$, and $\forall u \in P$, $d(u) = 1$.
- We can assume that $c = 1$, if we re-scale the given metric.
- Thus, I= $\min_{F' \subseteq P} \{|F'| + \sum_{u \in P} D(u, F')\}$

Preliminaries

- Let (P,D) be a metric with a point set $P = p_1, ..., p_n$. For any point $p_i \in P$, and $\forall r \geq 0$, we denote by $B(p_i, r)$ the set of points in P which are at distance at most r from p_i. For each i, $1 \leq i \leq n$, let $r_i > 0$ be the number satisfying $\sum_{p \in B(p_i, r_i)} (r_i - D(p_i, p)) = 1$.

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- We can easily see that $\forall i : 1 \leq i \leq n$, we have $\frac{1}{n} \leq r_i \leq 1$

Lemma 1

For every i, with $1 \leq i \leq n$, we have $\frac{1}{|B(p_i,r_i)|} \leq r_i \leq \frac{2}{|B(p_i,r_i/2)|}$.

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Proof

By the definition of r_i, we have $\sum_{p \in B(p_i,r_i)} (r_i - D(p_i, p)) = 1$, which implies $\sum_{p \in B(p_i,r_i)} r_i \geq 1$, and thus $r_i \geq 1/|B(p_i, r_i)|$. The other inequality follows directly from the following: $1 = \sum_{p \in B(p_i,r_i)} (r_i - D(p_i, p)) \geq \sum_{p \in B(p_i,r_i/2)} (r_i - D(p_i, p)) \geq |B(p_i, r_i/2)| * r_i/2$.

MP Algorithm

1. Compute the value of r_i for every $p_i \in P$.
2. Sort the input such that $r_1 \leq r_2 \leq \ldots \leq r_n$.
3. For $i = 1$ to n: if there is no open facility in $B(p_i, 2r_i)$ then open the facility at p_i.

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3. For $i = 1$ to n: if there is no open facility in $B(p_i, 2r_i)$ then open the facility at p_i.

This simple algorithm will return a set of open facilities for which the total cost is at most 3 times the minimum.

Cost Estimation

Lemma 2

$$\frac{1}{4}C_{OPT} \leq 4\sum_{p_i \in P} r_i \leq 6C_{OPT}.$$

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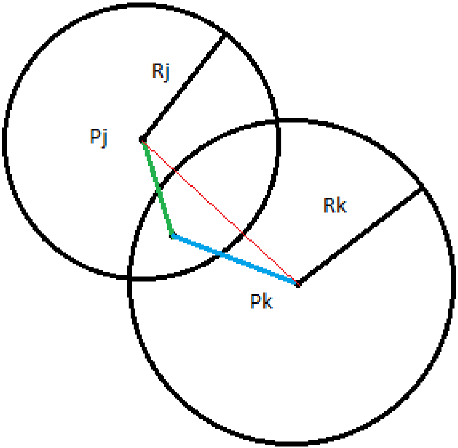
Proof:

Lower bound

- Since in the MP algorithm for every $p_i \in P$ there is an open facility within distance at most $2\ r_i$ (for if not, then the algorithm would open the facility at p_i), we get that $2\sum_{p_i \in P} r_i \geq C^c_{MP}$.
- It remains to show that $\sum_{p_i \in P} r_i$ is an upper bound for C^f_{MP}.

- We first observe that every $p_i \in P$ is contained in at most one ball $B(p_j, r_j)$, for some $p_j \in F_{MP}$.



So, $\displaystyle\sum_{p_i \in P} r_i \geq \sum_{p_j \in F_{MP}} \sum_{p_k \in B(p_j, r_j)} r_k.$

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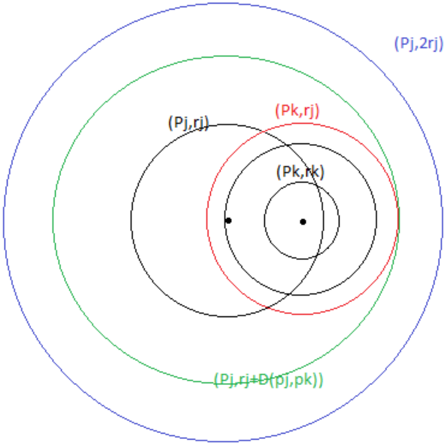
- Next, we observe that if $p_j \in F_{MP}$ and $p_k \in B(p_j, r_j)$, then we must have $r_j \leq 2r_k$.
- 

Lower bound

- $\sum_{p_i \in P} r_i \geq \sum_{p_j \in F_{MP}} \sum_{p_k \in B(p_j, r_j)} r_k \geq \sum_{p_j \in F_{MP}} \sum_{p_k \in B(p_j, r_j)} \frac{r_j}{2} = \frac{1}{2} \sum_{p_j \in F_{MP}} r_j |B(p_j, r_j)| \geq \frac{1}{2} \sum_{p_j \in F_{MP}} 1 = \frac{1}{2} C^f_{MP}$ (using Lemma 1)

Lower bound

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- Thus, we have: $2 * \sum_{p_i \in P} r_i \geq \frac{C^c_{MP}}{2} + \frac{C^f_{MP}}{2} \geq \frac{C_{MP}}{2} \geq \frac{C_{OPT}}{2}$

Estimating r_i in time $O(r_i n \log n)$

There is a constant factor approximation algorithm (randomized with high probability) of the above complexity.

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Let j_0 be the maximum integer j, with $1 \leq j \leq \log n$, such that $|B(p_i, 2^{-j})| \geq 2^j$. Then, we have $2^{-(j0+1)} \leq r_i \leq 2^{-j0+1}$.

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Proof

Use Lemma 1...

Algorithm

Our algorithm to estimate j_0 runs as follows:

- Set $j=\log n$.
- Decrease j by one until for the first time $:|B(p_i, 2^{-j})| \ge 2^j$

Approximation of $|B(p_i, 2^{-j})|$

- At each step, we pick uniformly at random, and with replacement, $K_j = c2^{-j} n \log n$ sample points
- Let N_j be the number of sample points that are inside the ball $B(p_i, 2^{-j})$.
- Return $\beta_j = nN_j/K_j$ as the estimator of $|B(p_i, 2^{-j})|$.

Quality of the estimator

Lemma 4

If $j \geq j_0 + 2$, then $\Pr[\beta_j \geq 2^j] < 1/\mathrm{poly}(n)$.

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Proof

Since $j \geq j_0 + 2$, it follows that $B(p_i, 2^{-j}) \subseteq B(p_i, 2^{-(j0+1)})$. Let q be the probability that a randomly chosen sample point is in $B(p_i, 2^{-j})$. We have $q \leq |B(p_i, 2^{-(j0+1)})|/n$. By the choice of j_0, we have $|B(p_i, 2^{-(j0+1)})| < 2^{j_0+1}$, and thus $q < 2^{j0+1}/n \leq 2^{j-1}/n$. The expected number of sample points that fall inside $B(p_i, 2^{-j})$ is $E[N_j] = qK_j < \frac{c\log n}{2}$. Applying the Chernoff bound, we obtain $\Pr[\beta_j \geq 2^j] = \Pr[N_j \geq c\log n] < 1/\mathrm{poly}(n)$.

Quality of the estimator

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Proof

Since $j \leq j_0 - 1$, it follows that $|B(p_i, 2^{-j})| \geq |B(p_i, 2^{-j_0})| \geq 2^{j_0} \geq 2^{j+1}$. We have that $q \geq 2^{j+1}/n$. The expected number of sample points that fall inside $B(p_i, 2^{-j})$ is $E[N_j] = qK_j \geq 2c\log n$. Applying the Chernoff bound, we obtain $\Pr[\beta_j \geq 2^j] = \Pr[N_j \geq c\log n] > 1 - 1/\text{poly}(n)$.

Lemma 6

The described procedure estimates the value of r_i to within a constant factor in time $O(r_i\, n \log n)$, with high probability.

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Proof

Let j_0' be the estimated value of j_0. By Lemmas 4 and 5, it follows that with high probability, $j_0 \leq j_0' \leq j_0 + 1$. If we use the value $r_i' = 2^{-j_0'}$ as an estimation of r_i, then by Lemma 3 we obtain that $r_i/4 \leq r_i \leq 2r_i$. Moreover, with high probability, the running time of the procedure is at most $\sum_{j=0}^{\log n} O(K_j) =$ $O(r_i n \log n)$.

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- To approximate the cost of the facility location problem it suffices to estimate the sum: $\sum_{i} r_i$ of the radii $r_1, ..., r_n$ of the points $p_1, ..., p_n$.

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- But, in order to guarantee that we get a constant factor approximation we need to sample $s = \Omega(n)$ points.
- This is due to the fact that the average radius can be as small as $1/n$.

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- If our sample is too small we double it and continue until we have found a sample of sufficient size.
- For the analysis we will parameterize the sample size s by the average value of the r_i. Combining this with the running time of the adaptive algorithm leads to a sublinear algorithm.

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- E[time] = s*E[one step] =s$*$ E[one step] = $\mathrm{sO}(\frac{1}{\mathrm{n}}\sum_{\mathrm{i}} \mathrm{r_i nlogn}) = \mathrm{O}(\mathrm{nlog^2 n})$.

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- $E[X_i] = \frac{\sum_j r_j}{n}$

- Let $S = \sum_{i=1}^{s} x_i$ and hence,
  $E[S] = \frac{s * \sum_i r_i}{n} = \frac{\Theta(\frac{n}{c} \log n) \sum_i r_i}{n} = \Theta(\frac{\sum_i r_i}{c} * \log n) = \Theta(\log n)$.

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- S will be used as an estimator of $\frac{s}{n} \sum_i r_i$

- Let $S = \sum_{i=1}^{s} x_i$ and hence, $E[S] = \frac{s * \sum_i r_i}{n} = \frac{\Theta(\frac{n}{c} \log n) \sum_i r_i}{n} = \Theta(\frac{\sum_i r_i}{c} * \log n) = \Theta(\log n)$.
- S will be used as an estimator of $\frac{s}{n} \sum_i r_i$
- From Hoeffding inequality and $0 \le x_i \le 1$:
  - $\Pr[S \ge (1+\varepsilon)E[S]] \le e^{-\frac{\varepsilon^2 E[S]}{2(1+\varepsilon/3)}}$
  - $\Pr[S \ge (1-\varepsilon)E[S]] \le e^{-\frac{\varepsilon^2 E[S]}{2}}$
  - $\Pr[|S - E[S]| \ge \varepsilon E[S]] \le 2e^{-\Theta(\varepsilon^2 E[S])} - 2e^{-\Theta(\varepsilon^2 \log n)}$

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Algorithm

- We start in the first phase by guessing $c = n$ (underestimation of the cost).
- If $S < \frac{s}{n}c$, then we start a new phase with estimated cost $c/2$, and so on.
- If $S \geq \frac{s}{n}c$, we return S*n/s as the approximation of the cost.

- The probability that the algorithm ends in a bad phase (when S far away from $\frac{s}{n} * c$ is low, because $\Pr[S \geq (1+\varepsilon) * E[S]] <$1/poly(n), as shown above.

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Theorem

There exists a constant factor approximation algorithm for the uniform case of the Minimum Facility Location problem which runs in time $O(n\log^2 n)$ with high probability.

Lower bounds

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Theorem 2

For any $\varrho \geq 1$, every approximation algorithm (even a randomized one) with approximation ratio ϱ for the cost of the Minimum Facility Location problem as defined above requires time $\Omega(n^2)$.

Proof

We show the existence of two instances of the metric spaces which are undistinguishable by any o(n^2)-time algorithms and such that the cost of the Minimum Facility Location in one instance is greater ϱ times than the one in the other instance (for every ϱ).

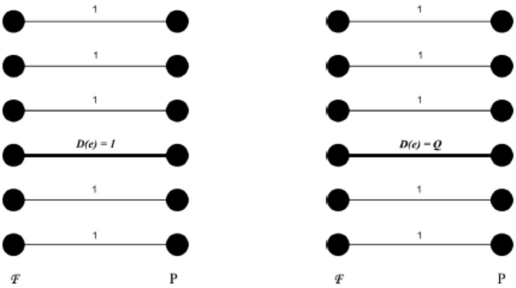

Fig. 1. Two metric spaces undistinguishable by any $o(n^2)$-time algorithms whose costs of the Minimum Facility Location differ by factor ϱ. The perfect matching connecting $\mathcal{F}$ with P is selected at random and the edge e is selected as a random edge from the matching. We set $Q = 2\,n\,(\varrho - 1) + 2$. The distances not shown are all equal to $n^3\,\varrho$