GIAN Course on Distributed Network Algorithms

The Power of Locality

Case Study: Graph Coloring

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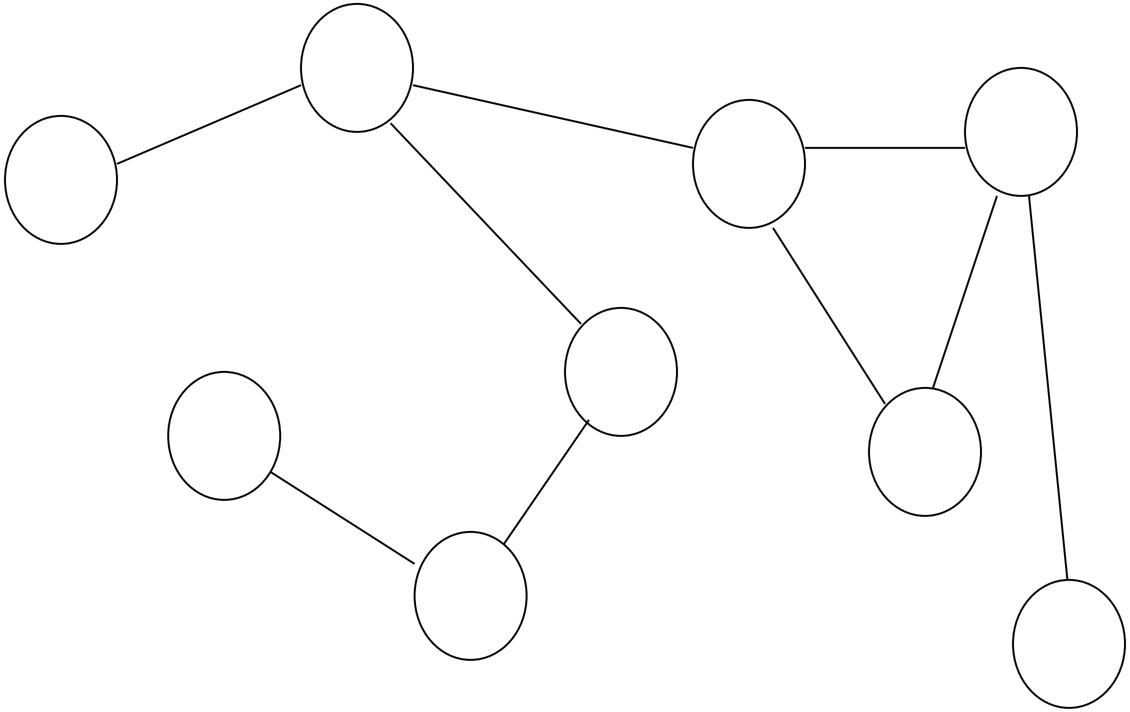
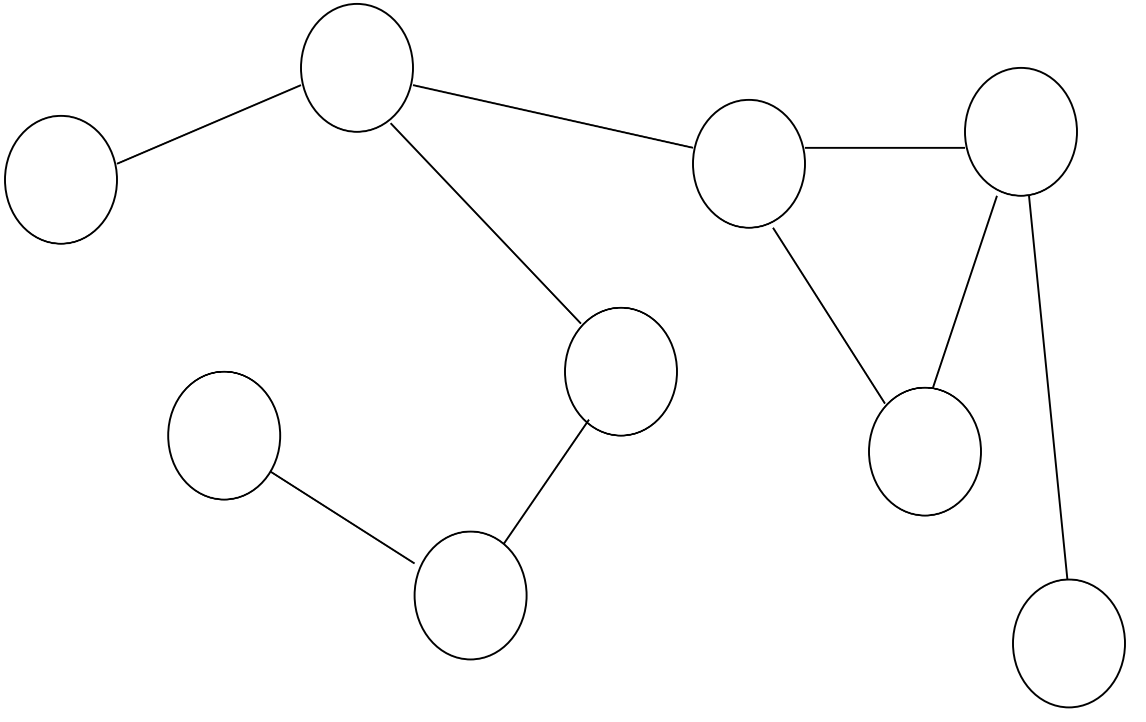
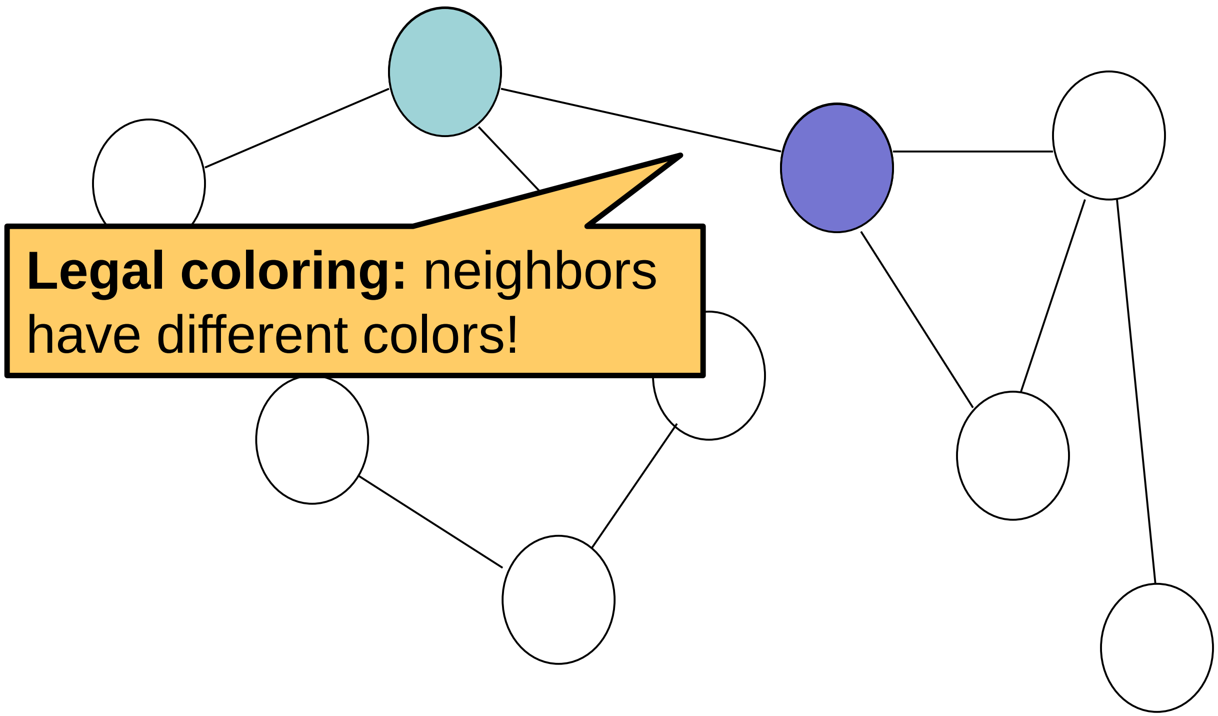


Case Study: Graph Coloring

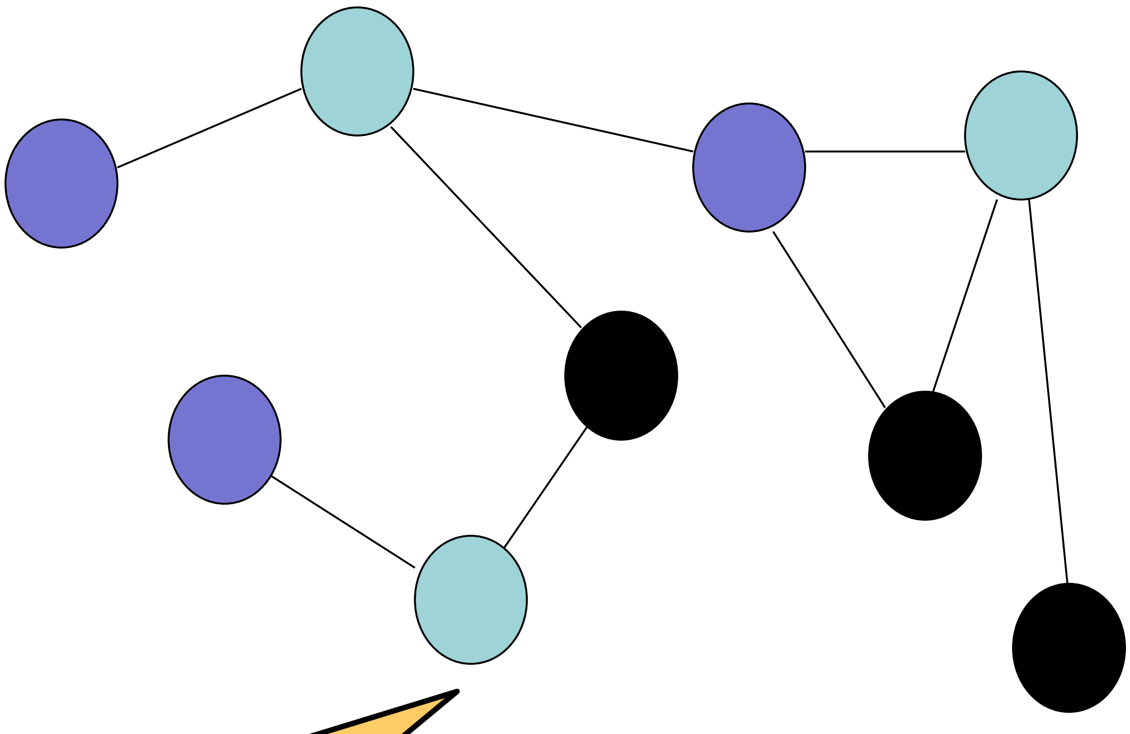
Assign colors to nodes.



Case Study: Graph Coloring



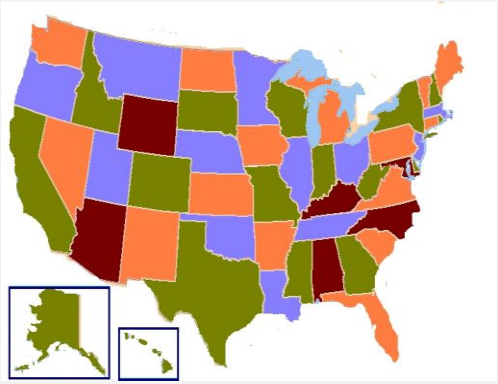
Case Study: Graph Coloring



Optimal coloring: Minimal number of colors (aka chromatic number)

Applications

Country Maps

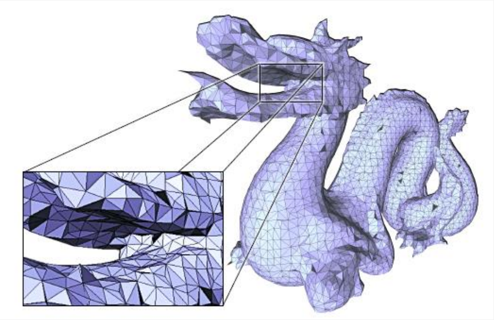


- Neighboring states should have different colors!
- Famous 4-color theorem: any map can be painted with four colors!

Medium Access

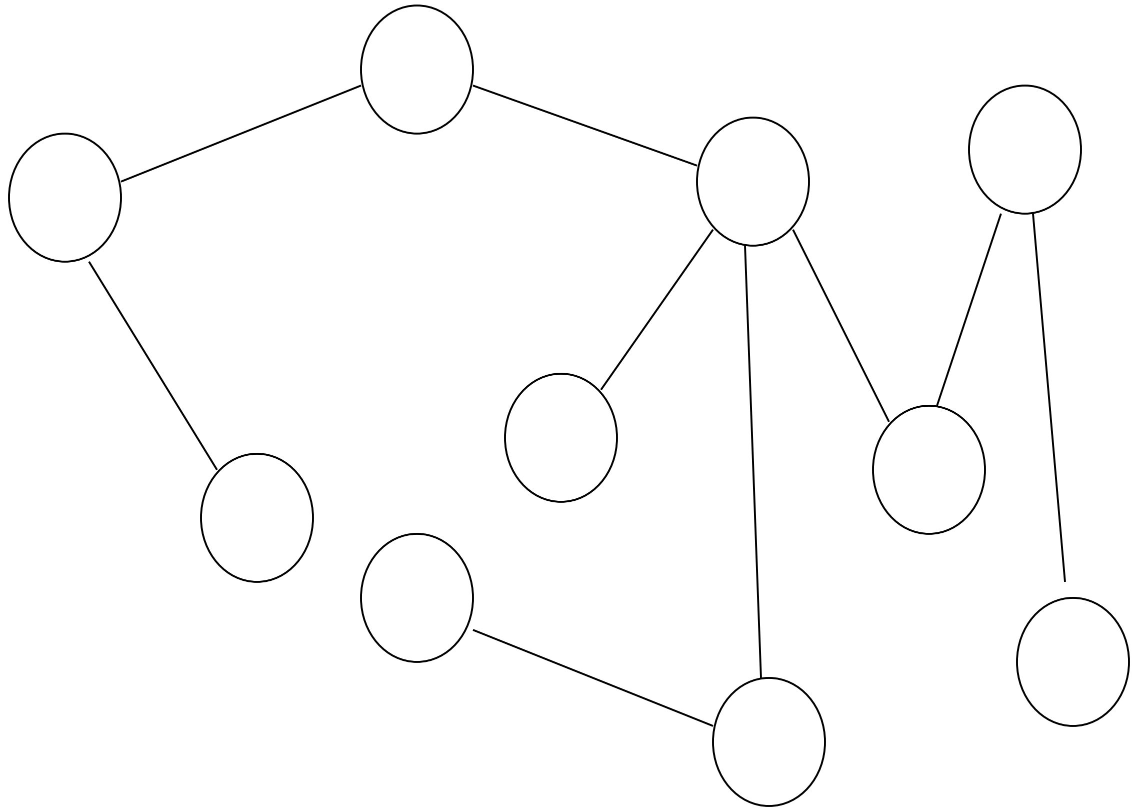
- Interference-free, efficient utilization of spectrum
- Neighboring cells should have different frequencies!
- Colors = frequencies, channels, etc.

Image Processing



- Chromatic scheduling for physical simulation
- Process nodes of same color in parallel without determinacy race
- No coordination, no mutual exclusion needed

Legal coloring? Chromatic number?



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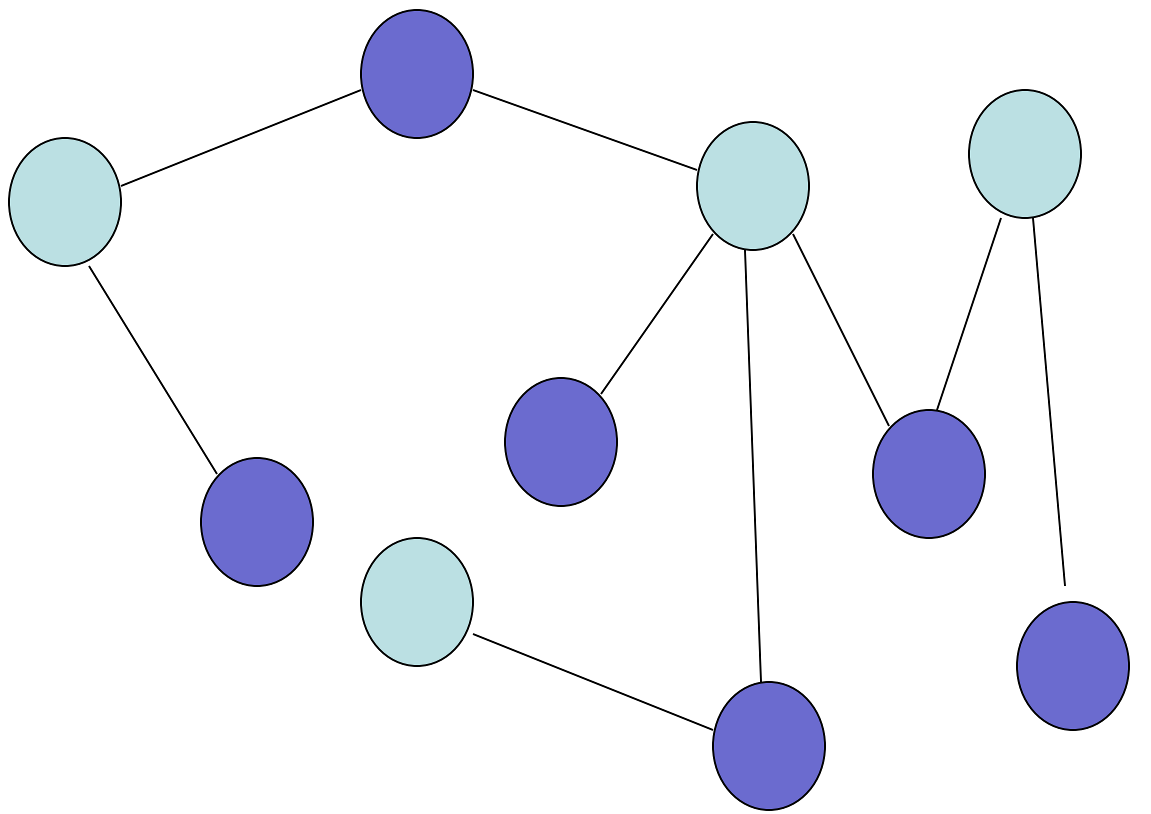
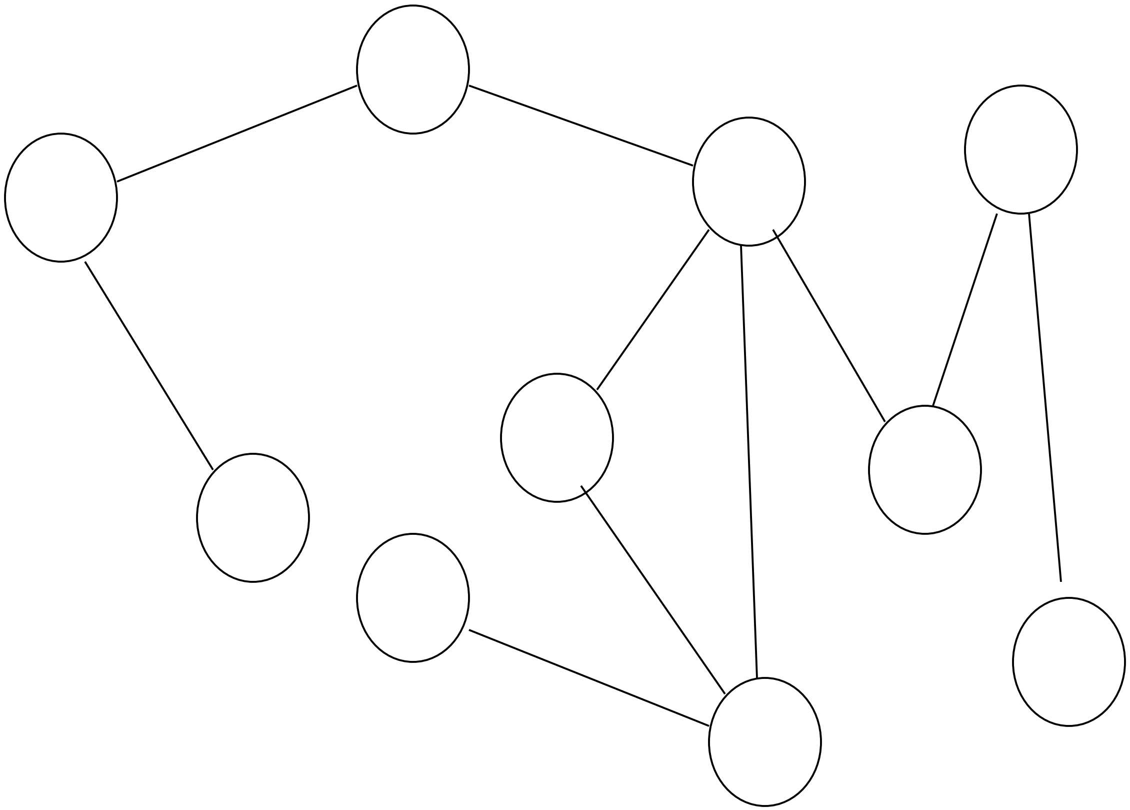


Tree! 2 colors are enough...

What about this example?



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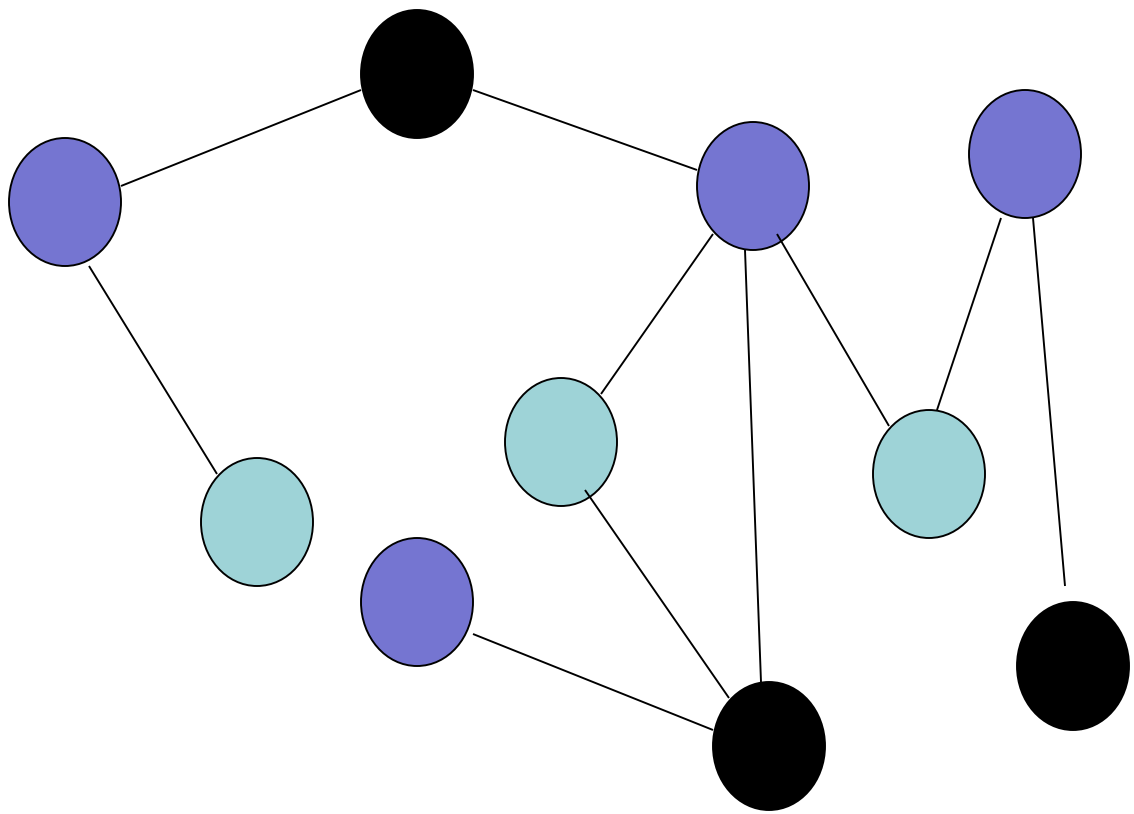


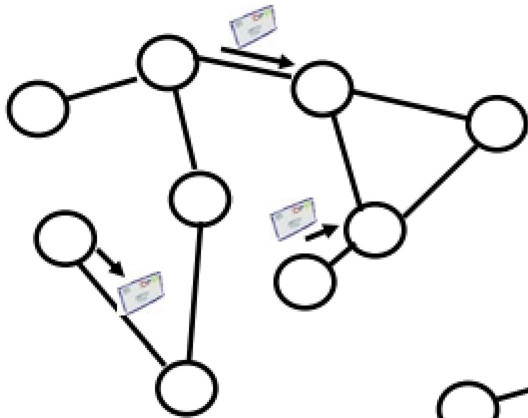
3 colors needed and enough...

How to color a graph in a distributed manner?

How to color a graph in a distributed manner?

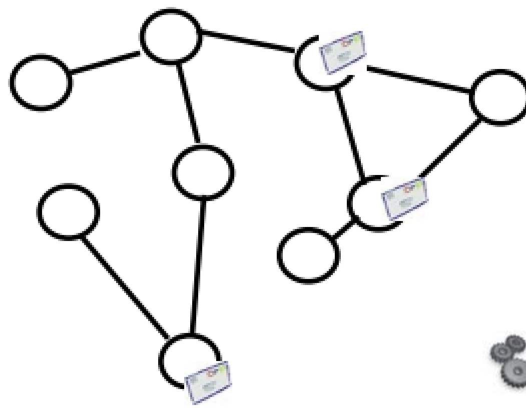
The LOCAL Model: A Convenient Synchronous Model

Send...

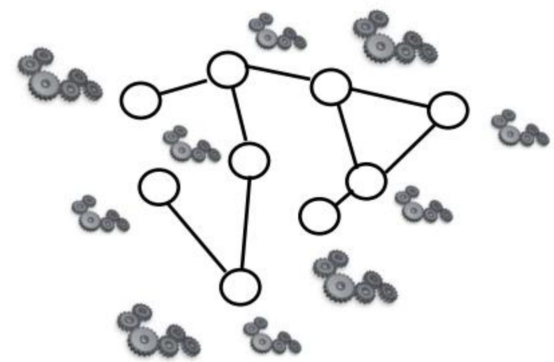


... in each round!

... receive...



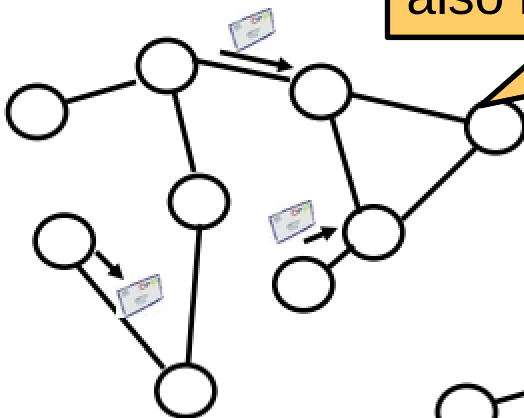
... compute.



The LOCAL Model: A

We will see in this course: there are techniques to execute an algorithm designed in the simple LOCAL model also in asynchronous networks!

Send...



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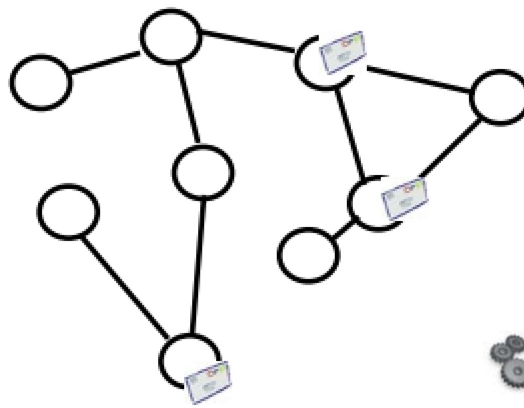

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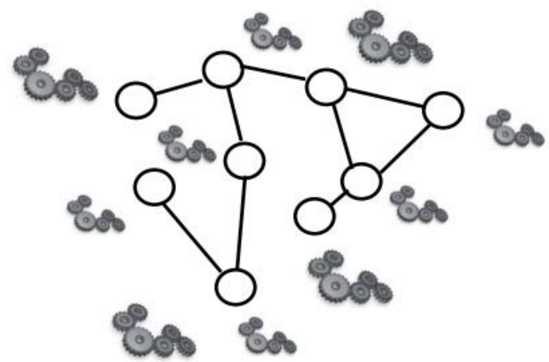

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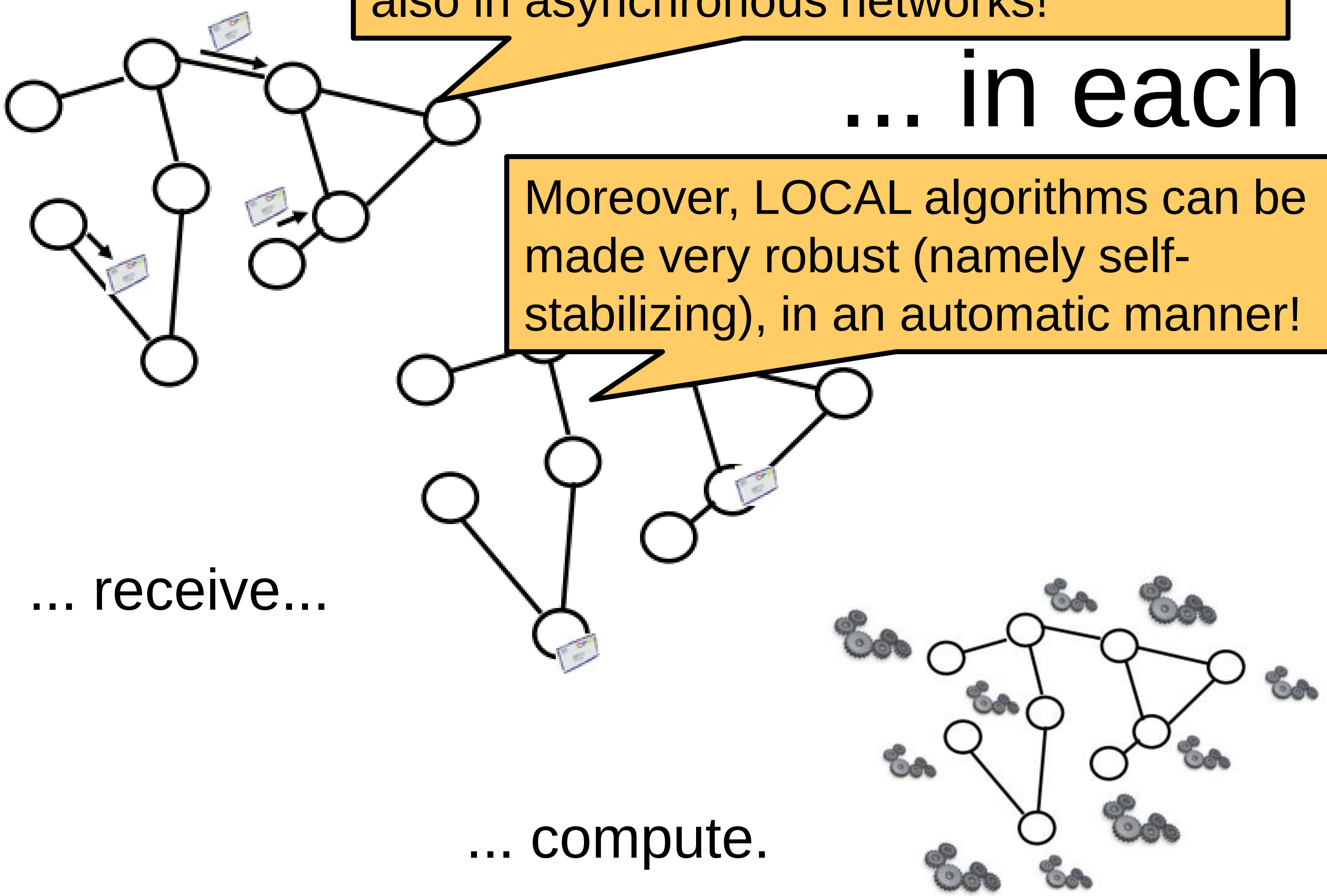
Moreover, LOCAL algorithms can be made very robust (namely self-stabilizing), in an automatic manner!



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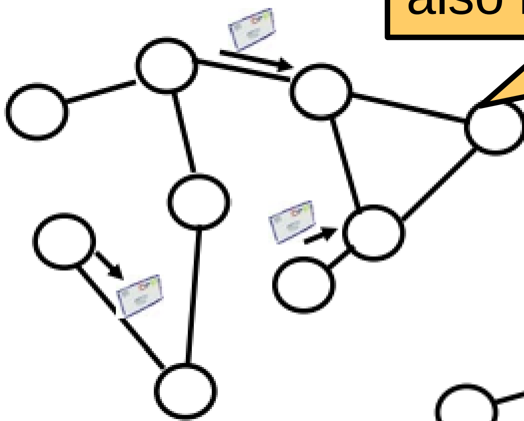

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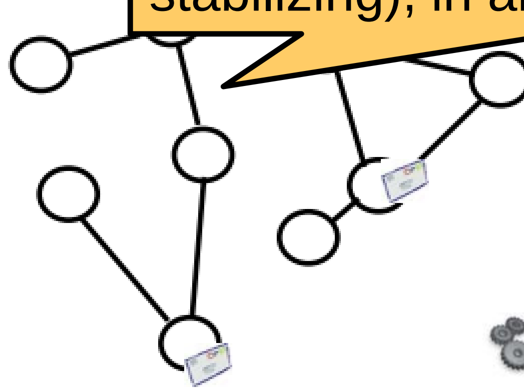

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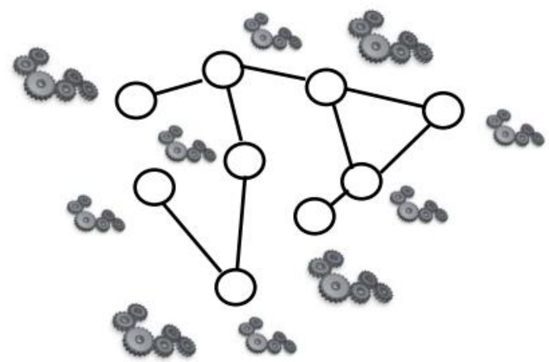


Unlike CONGEST model: message size and link capacity not bounded.

... compute.

LOCAL Performance Metrics

- ❑ **Time Complexity:**

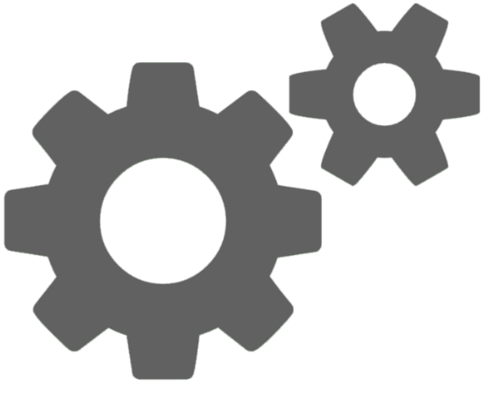
  Number of communication rounds

- ❑ **Message Complexity:**

  Number of messages sent

- ❑ **Local Computation:**

  Complexity of local computations

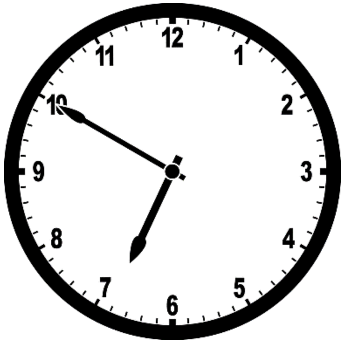


LOCAL Performance Metrics

What else?

- **Time Complexity:**

  Number of communication rounds



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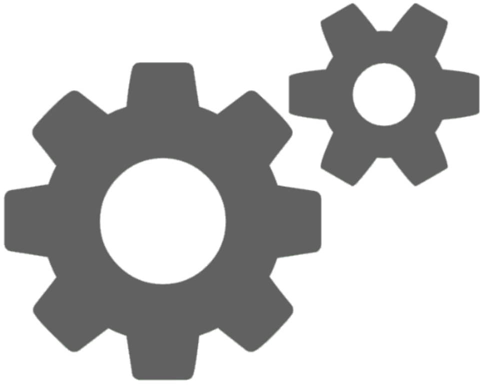


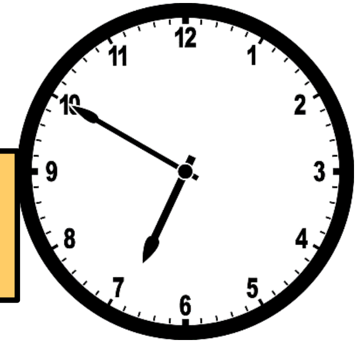
LOCAL Performance Metrics

What else?

❑ **Time Complex**

Num

Quality of solution: Approximation ratio for example („price of locality“).



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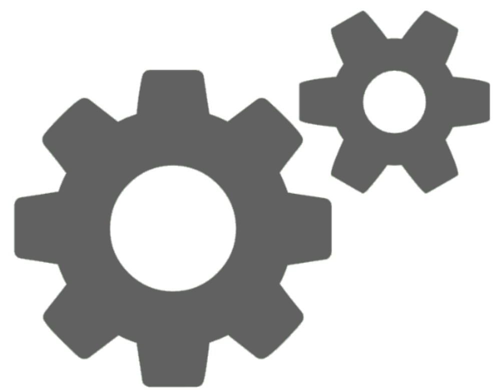
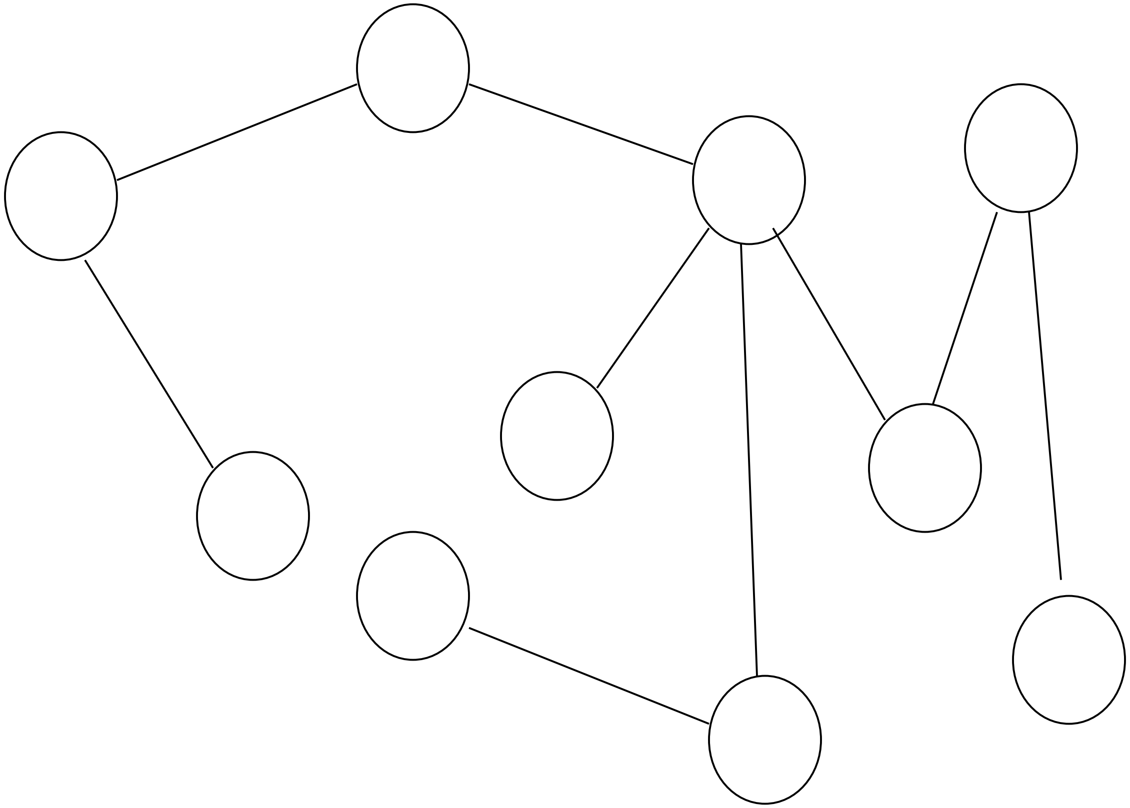


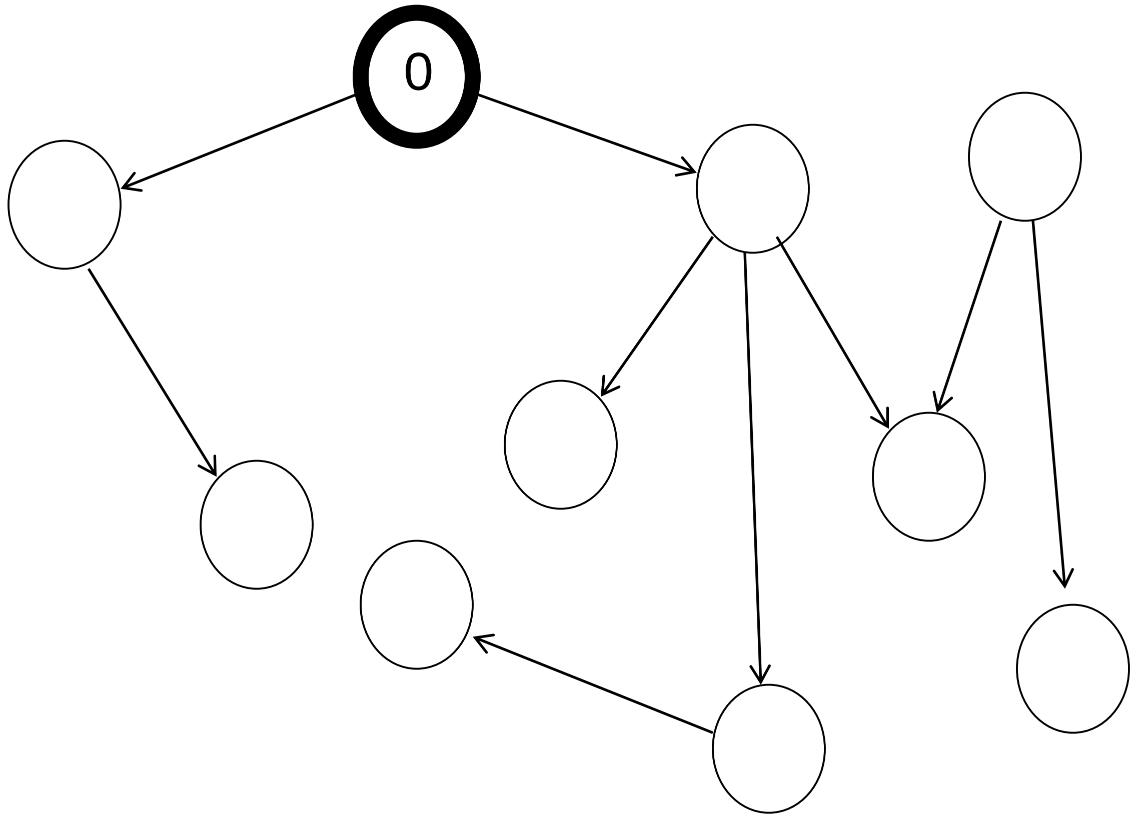
How to color a *tree* in a distributed manner?



How to color a *rooted tree* in a distributed manner?

Simplification:

- ❑ Assume unique node IDs
- ❑ Assume rooted
- ❑ Root ID 0



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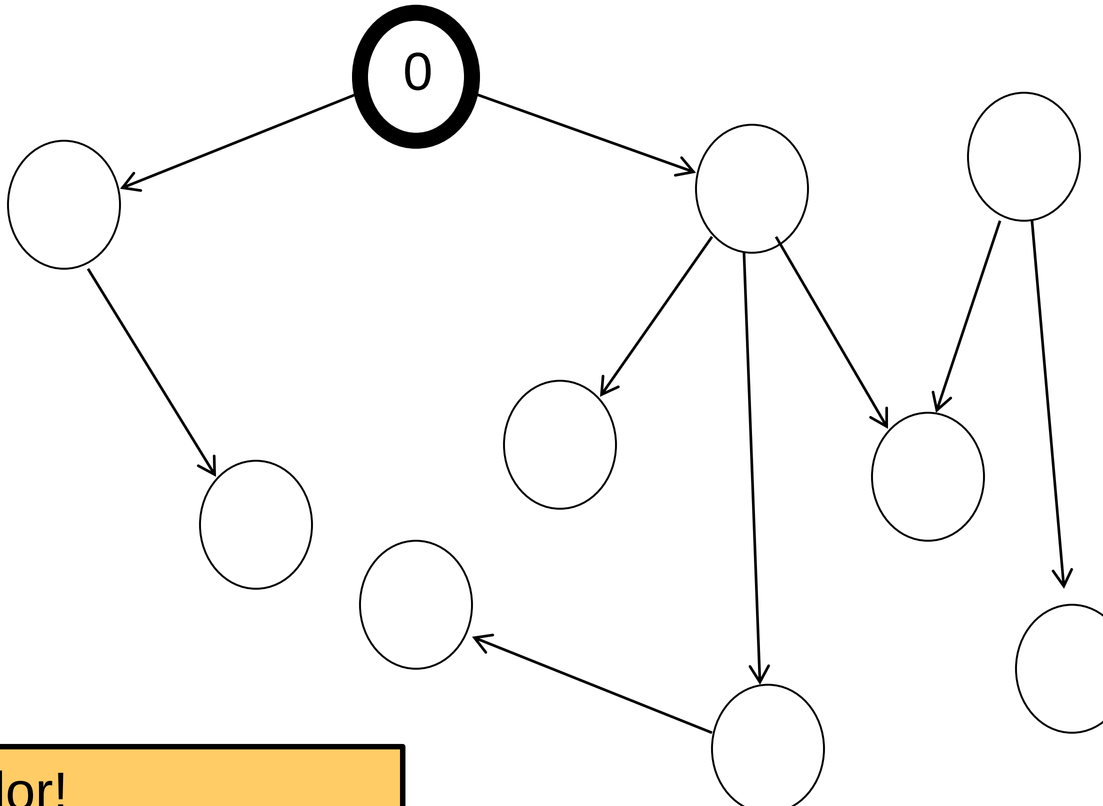
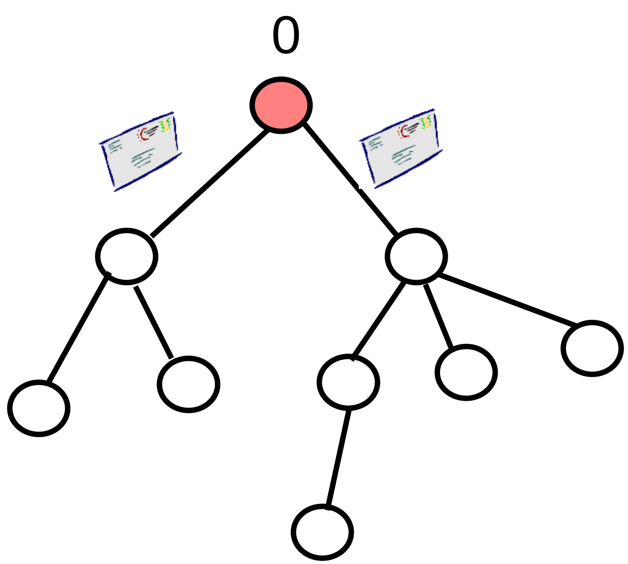


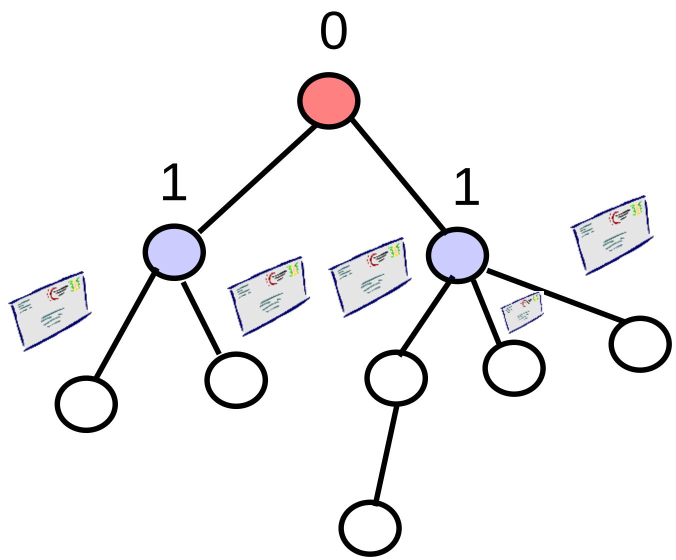
Idea: interpret ID as color!
Communicate my color to children and take opposite color from my parent!

Slow Distributed Tree Coloring: Example



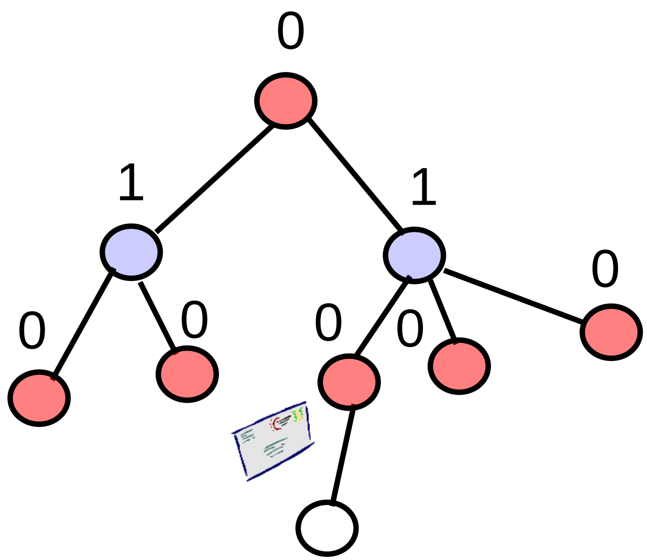
Round 1

Slow Distributed Tree Coloring: Example



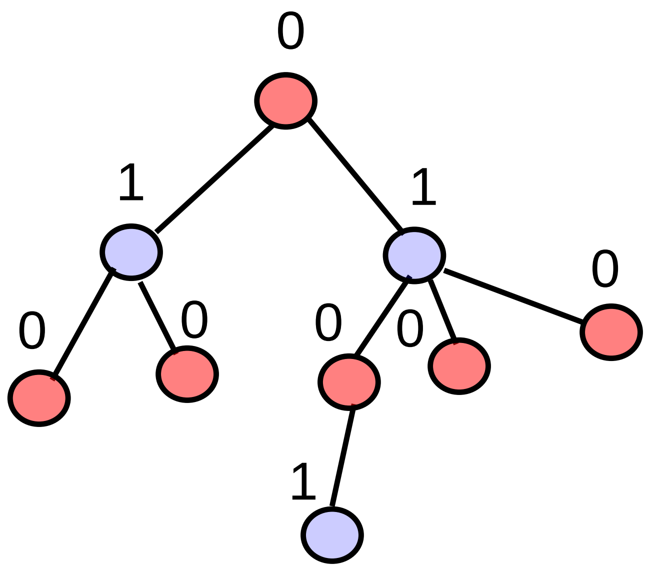
Round 2

Slow Distributed Tree Coloring: Example



Round 3

Slow Distributed Tree Coloring: Example



Round 3

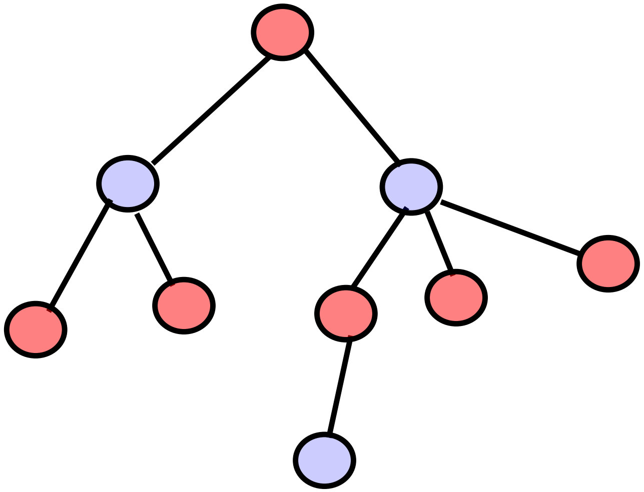
Slow Tree Algo

If root: color 0, send 0 to children

Otherwise: each node *v*:

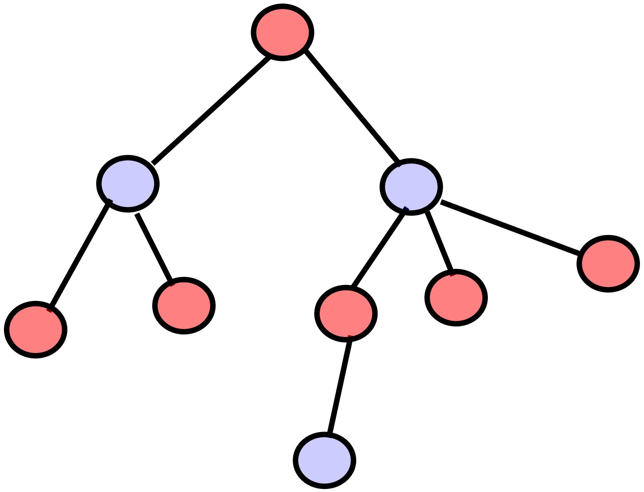
- Wait for message *x* from parent
- Choose color *y=1-x*
- Send *y* to children

Slow Tree: Analysis



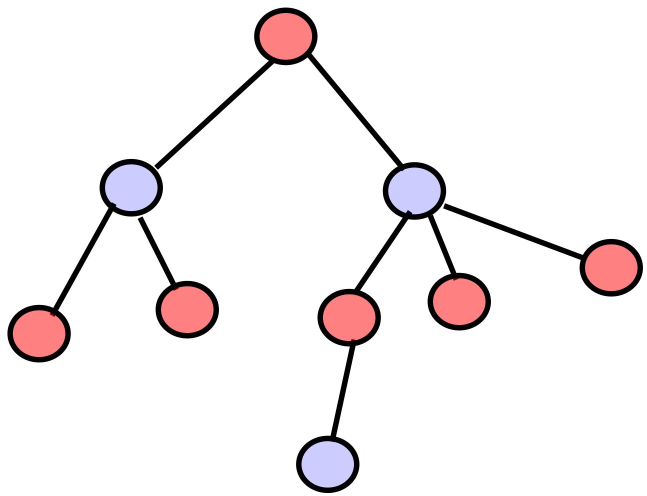
- ❑ **Approximation quality:**
- ❑ **Time complexity:**
- ❑ **Message complexity:**
- ❑ **Local complexity:**

Slow Tree: Analysis



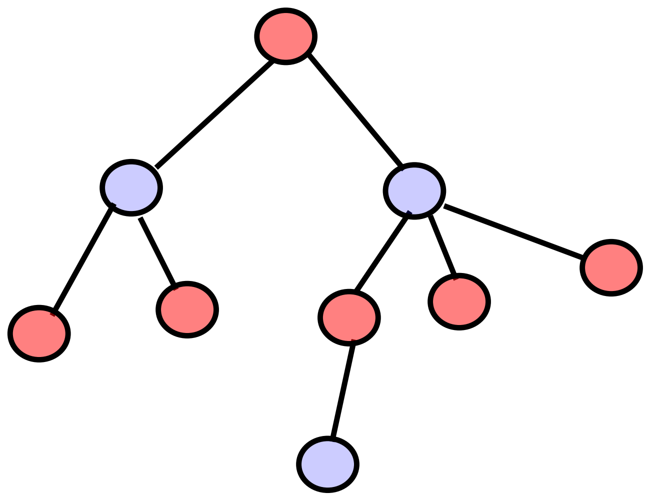
- **Approximation quality:** # colors?
- **Time complexity:** # rounds?
- **Message complexity:** # messages?
- **Local complexity:** local computations?

Slow Tree: Analysis



- **Approximation quality:** 2 colors suffice!
- **Time complexity:** O(n), depth of the tree
- **Message complexity:** O(n)
- **Local complexity:** trivial, just flip!

Slow Tree: Analysis

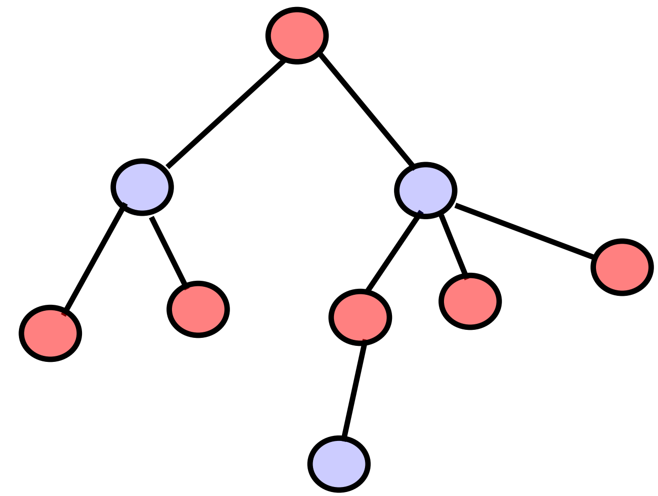


Can we do faster?

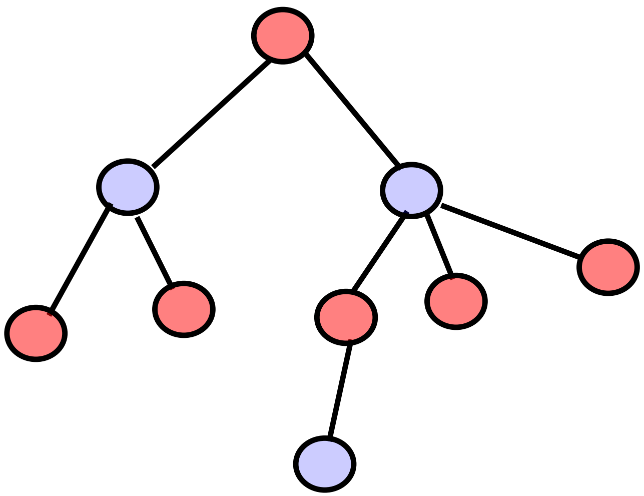
- **Approximation quality:** 2 colors suffice!
- **Time complexity:** O(n), depth of the tree
- **Message complexity:** O(n)
- **Local complexity:** trivial, just flip!

Ultra Fast Distributed Tree Coloring

- Yes we can!
- 3-coloring in O(log* n) rounds



Ultra Fast Distributed Tree Coloring



- Yes we can!
- 3-coloring in O(log* n) rounds
- Idea: based on ID manipulations
  - Again: interpret ID as color

Unique IDs → legal (but expensive) coloring!
How can we quickly reduce the ID space?

Intuition: n vs log* n

log n: How many times do I have to **:2** until <2?

n, n/2, n/4, n/8, …, 8, 4, 2, 1

log n

Intuition: n vs log* n

log n: How many times do I have to **:2** until <2?

$$n, n/2, n/4, n/8, \ldots, 8, 4, 2, 1$$

log n

loglog n: How many times do I have to √x until <2?

$$n, \sqrt{n}, \sqrt{\sqrt{n}}, \sqrt{\sqrt{\sqrt{n}}}, \ldots, <2$$

loglog n

Intuition: n vs log* n

log n: How many times do I have to **:2** until <2?

n, n/2, n/4, n/8, …, 8, 4, 2, 1

log n

loglog n: How many times do I have to √x until <2?

n, √ n, √ √ n, √ √ √ n, …, <2

loglog n

log* n: How many times do I have to log x until <2?

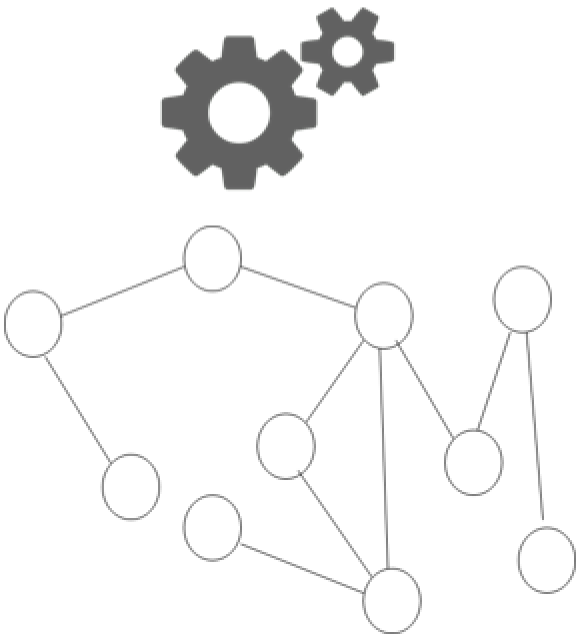
n, log n, loglog n, logloglog n, …, <2

log* n


n = atoms in universe ≈ 10^80
log*(atoms in universe) ≈ 5

Slow Algo

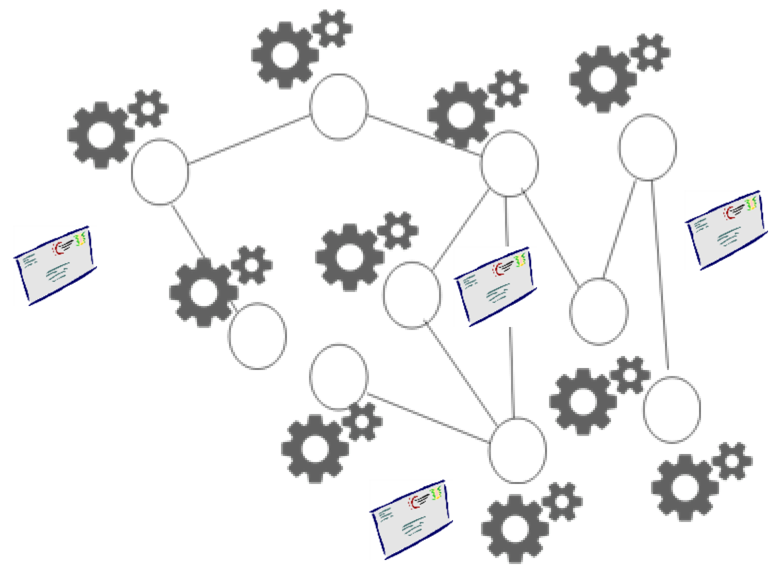
No parallelism!



Time: n

Fast Algo

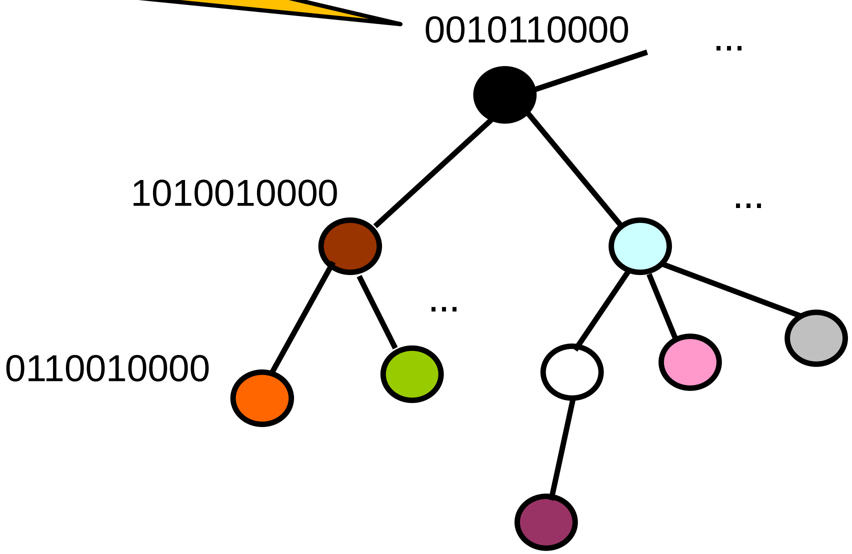
Efficient parallel manipulations!



Time: log* n

Log*-Time Coloring with Label Manipulation

Initially ID = label of node *v* =color c_v

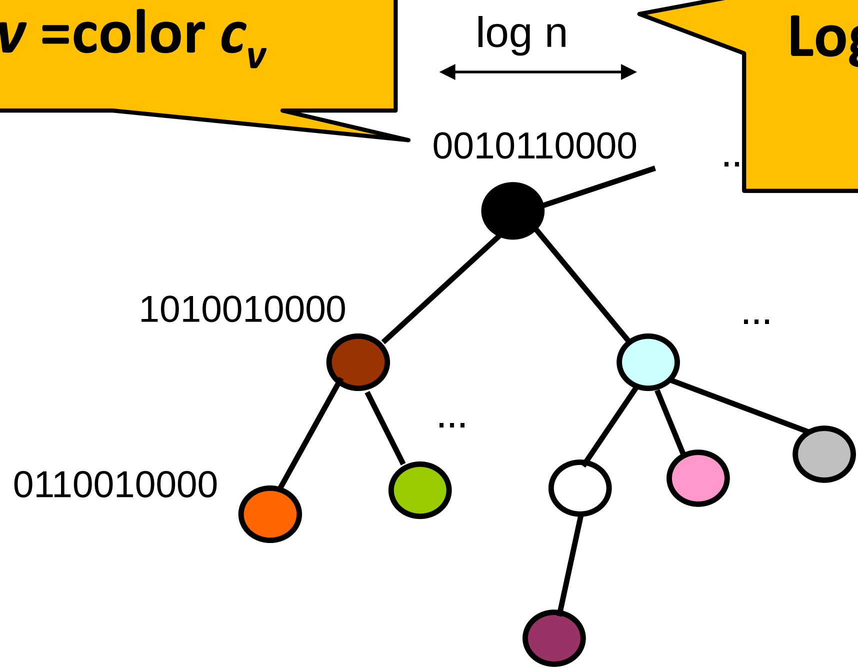


initially

Log*-Time Coloring with Label Manipulation

Initially ID = label of node *v* =color c_v

Log n bits to represent n unique IDs



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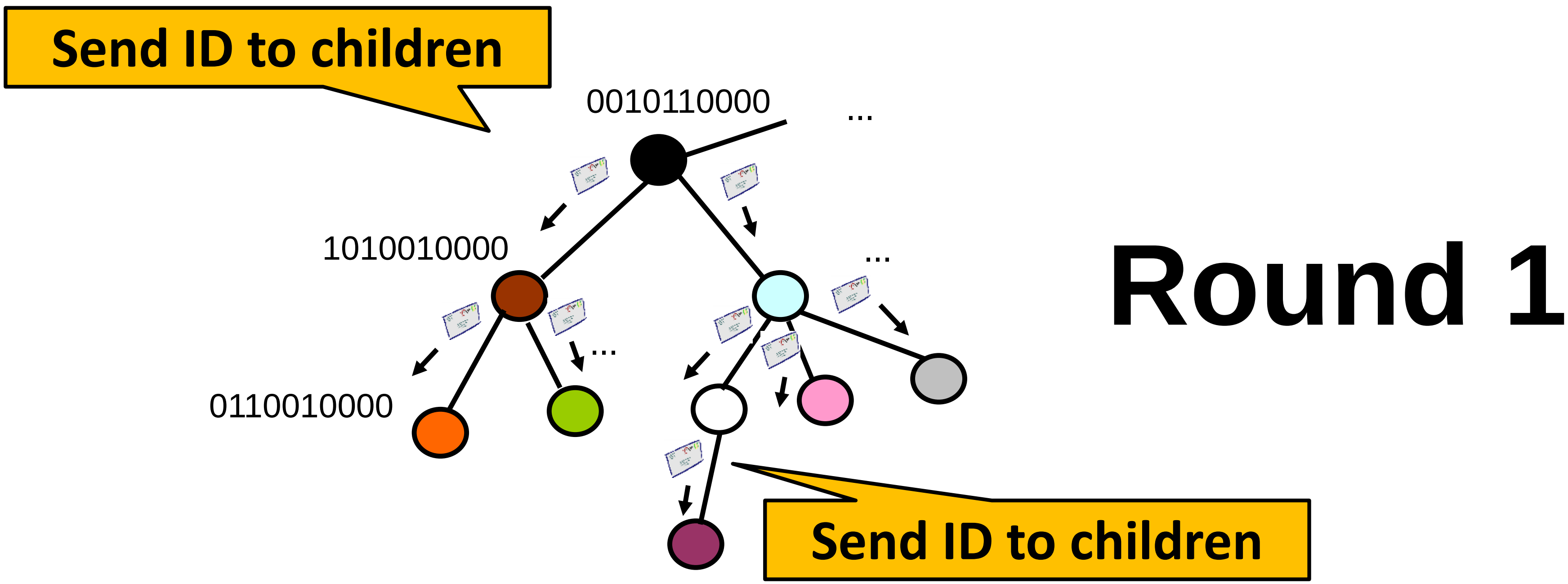
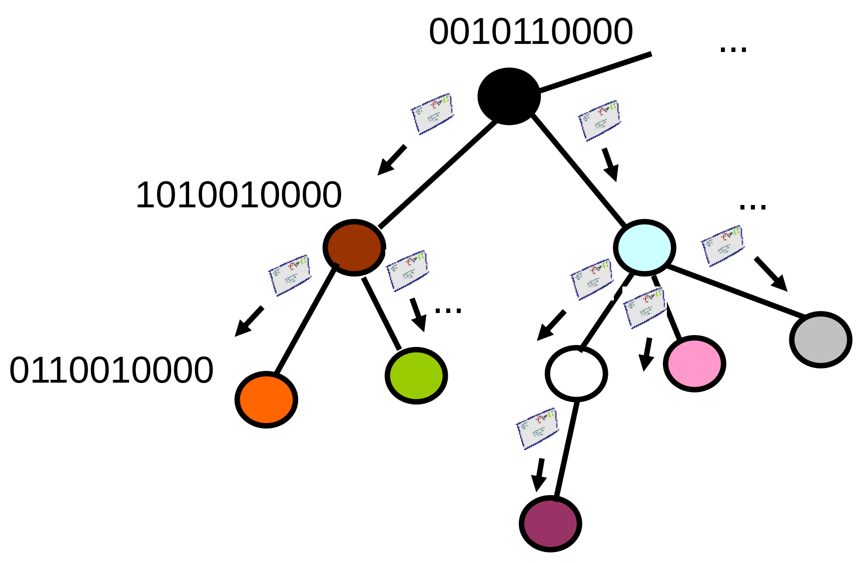
Log*-Time Coloring with Label Manipulation



Algorithm: in round i, node v:

1. Send my c_v to children (in parallel!)
2. Receive parent ID/color c_p

Log*-Time Coloring with Label Manipulation



Round 1

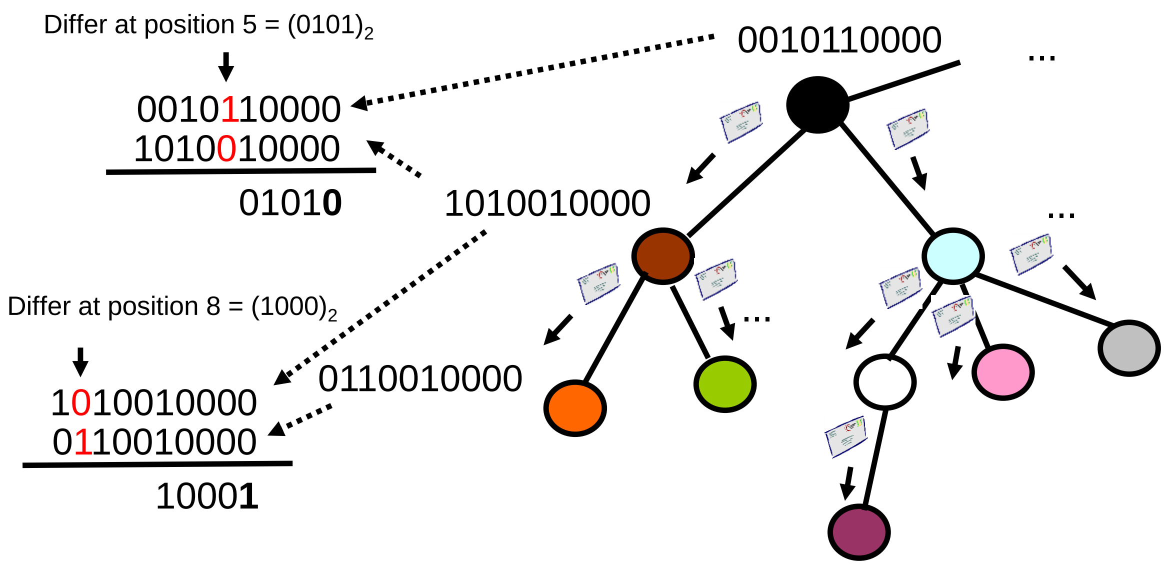
Algorithm: in round i, node v:

1. Send my c_v to children (in parallel!)
2. Receive parent ID/color c_p
3. Let i be the smallest index where c_v and c_p differ (from right, binary)
4. My new c_v = i || c_v(i)

ID = color for next round: the position!

Log*-Time Coloring with Label Manipulation

Example:



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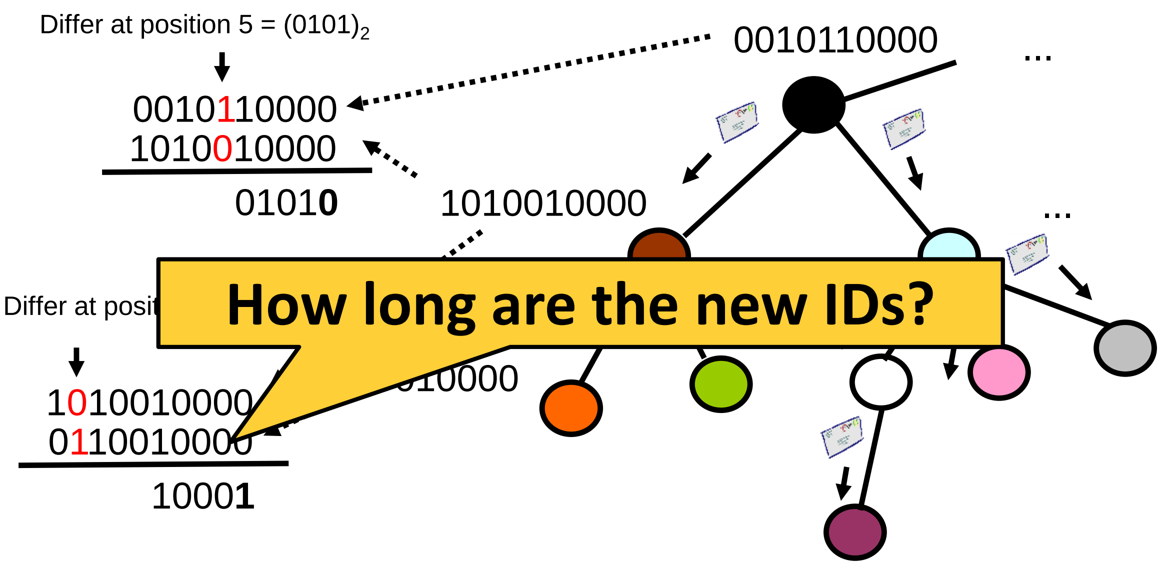

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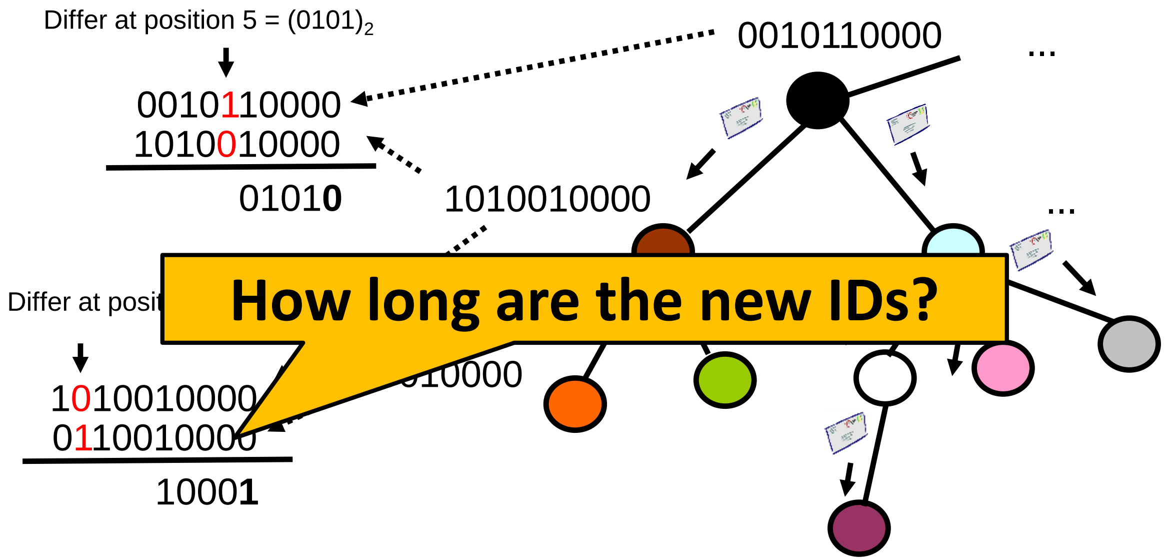

How long are the new IDs?

Round 1

Describing position in x-bit string takes log x bits, so: loglog n bits

allel!)

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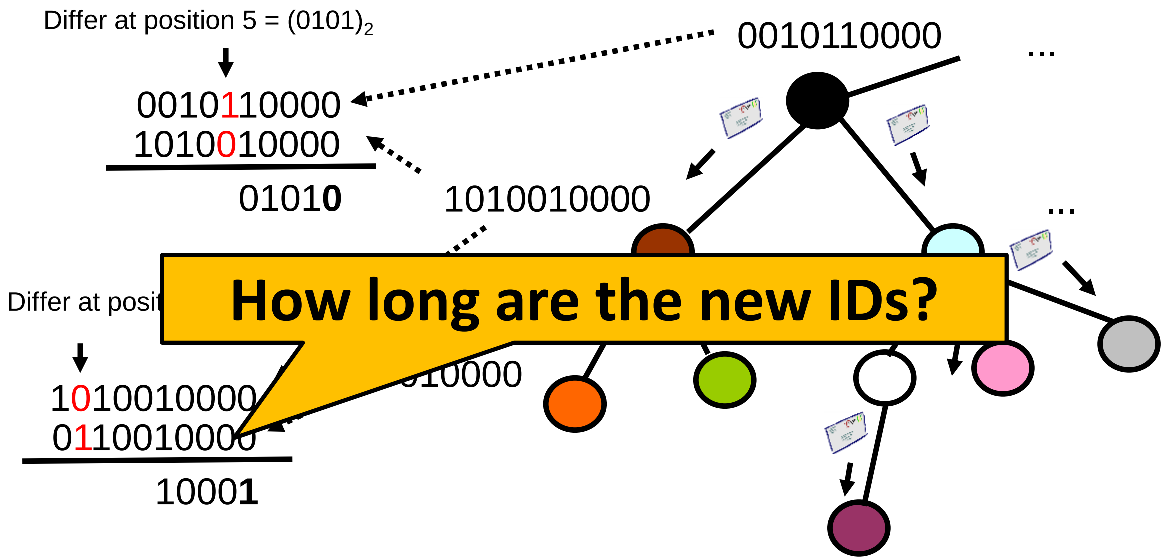
Log*-Time Coloring with Label Manipulation



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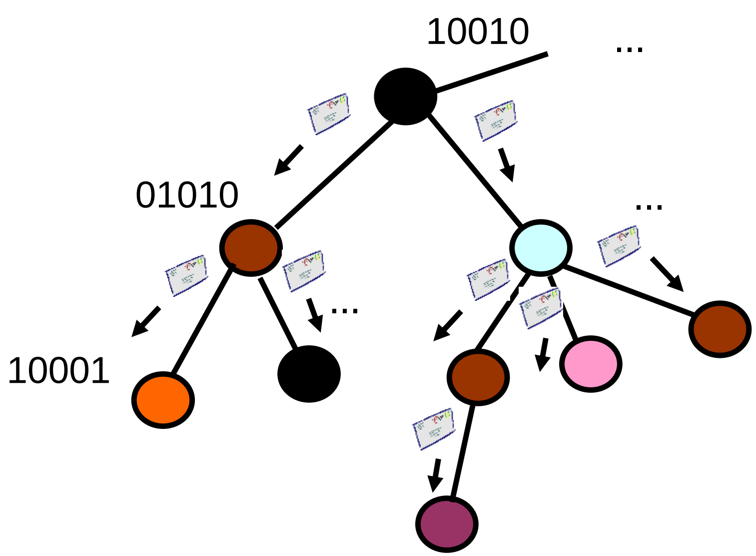
+1 bit

Log*-Time Coloring with Label Manipulation



Round 2

Algorithm: in round i, node v:

1. Send my c_v to children (in parallel!)
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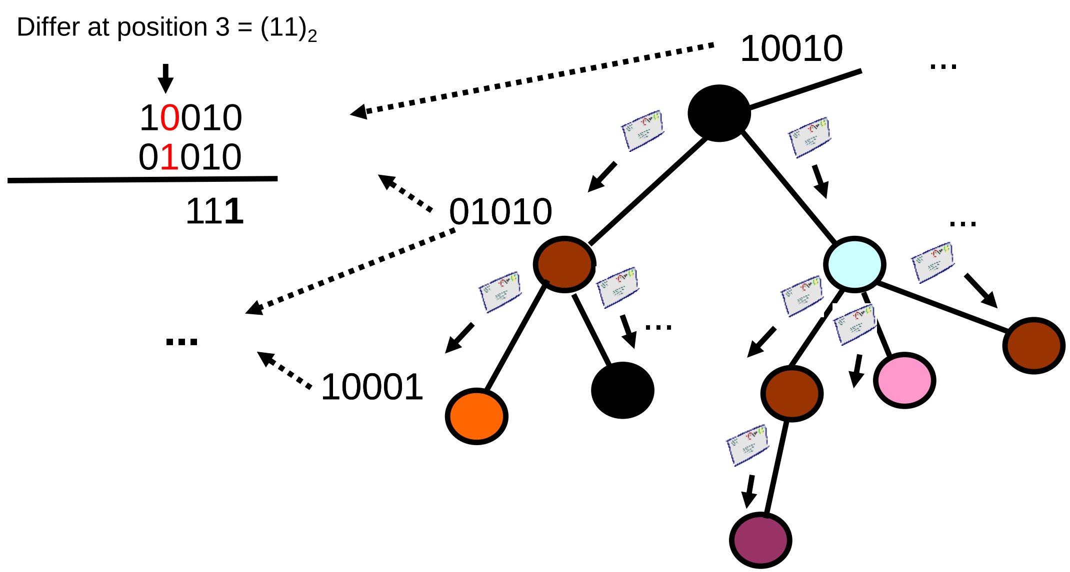
Log*-Time Coloring with Label Manipulation



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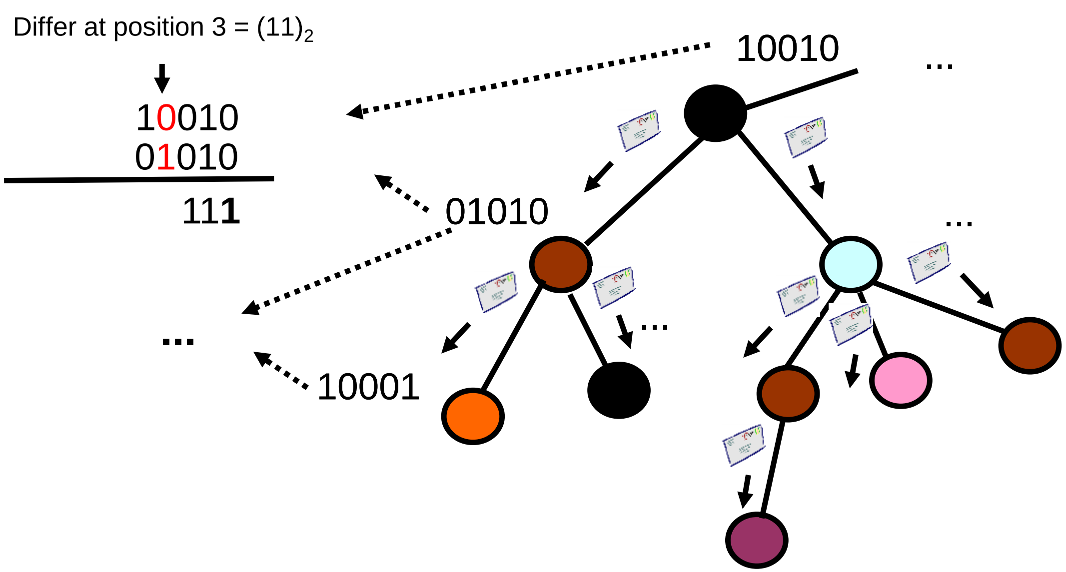

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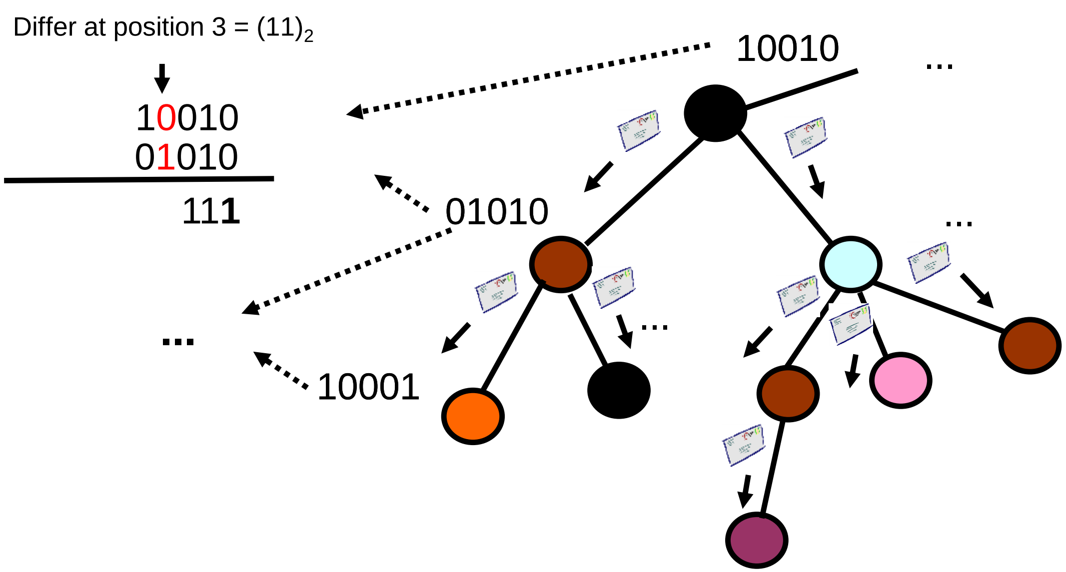

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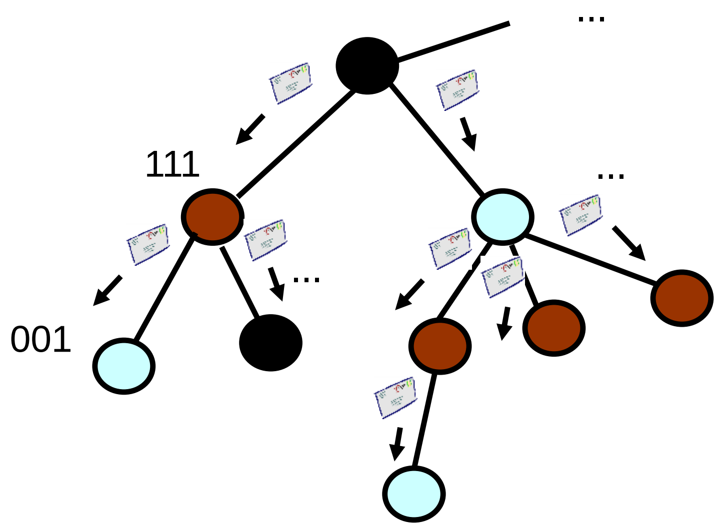
+1 bit

Log*-Time Coloring with Label Maniplulation



Round 3

Algorithm: in round i, node v:

1. Send my c_v to children (in parallel!)
2. Receive parent ID/color c_p
3. Let i be the smallest index where c_v and c_p differ (from right, binary)
4. My new c_v = i || c_v(i)

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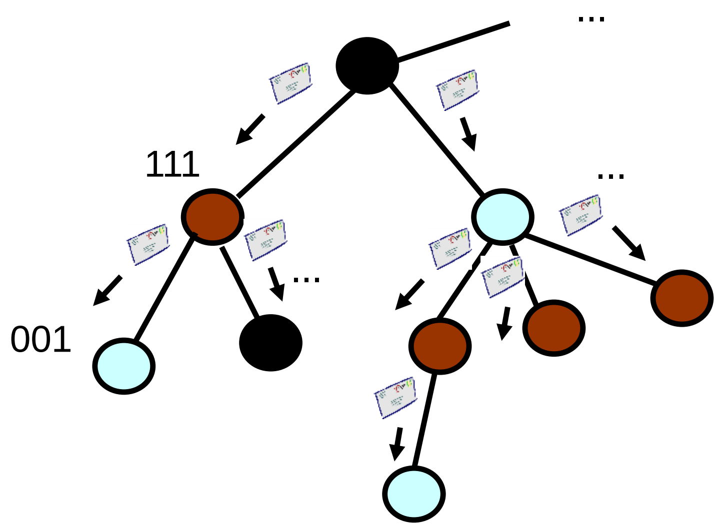

etc.!

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  - The definition of log*!

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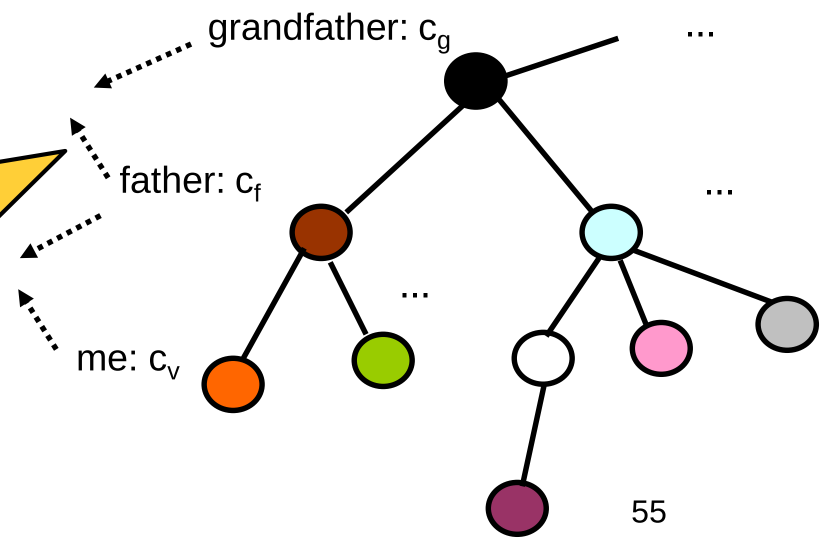
Algorithm: My new c_v = i || c_v(i)

By contradiction: To get the same ID as my father, I need to differ at same position from father as father from grandfather. But then last bit must be different: there I took my own bit (and father will do the same with his different bit)!



Summary of Algorithm

6-Colors

Assume: legal initial coloring, root with label c_v=0
Each other node v does (in parallel):

Send c_v to kids

Repeat (until $c_w \in \{0,...,5\}$ for all w):

1. *Receive* c_p from parent
2. Interpret c_v/c_p as little-endian bitstrings c(k)...c(1)c(0)
3. Let i be smallest index where c_v and c_p differ
4. New label is: **$i||c_v(i)$**
5. *Send* c_v to kids

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Note: we stop if color in {0,…,5}: why?

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But actually colors 110 (for color „6") and 111 (for color „7") are not needed, as we can do another round! IDs of three bits can only differ at positions 00 (for „0"), 01 (for „1"), 10 (for „2")

With 6-COLORS algorithm we can get down to 6 colors.

What about improving it to 2 colors?

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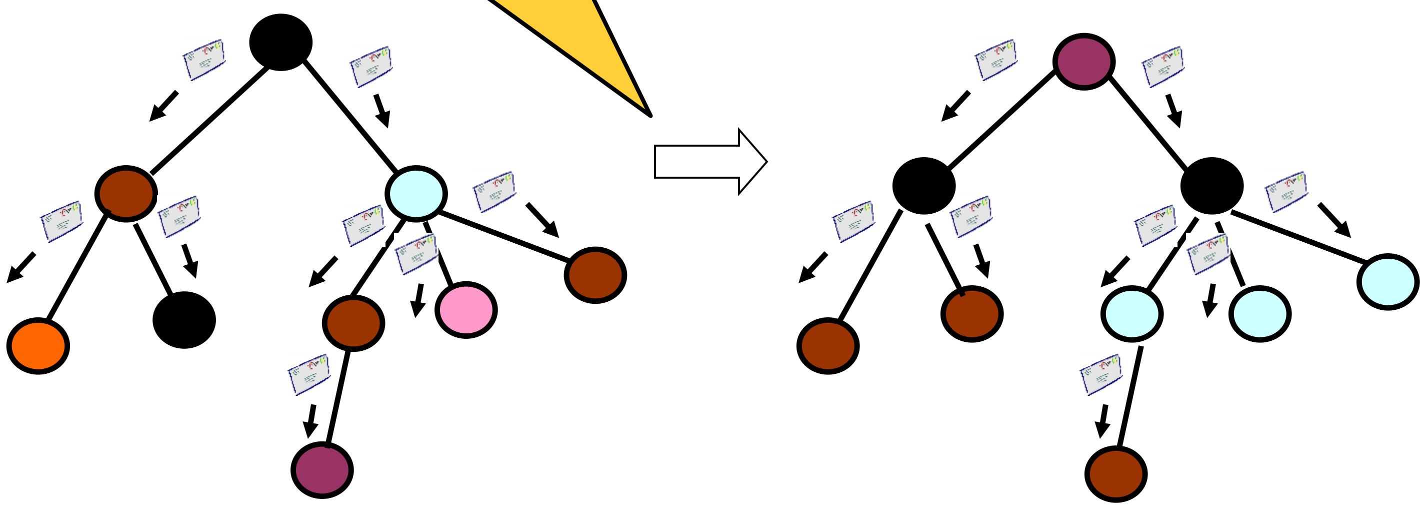
Impossible: takes linear time.
What about 3 colors?

Observation: Shift Down

Let us note a simple trick: shift colors down by one level makes siblings „independent". And preserves legal coloring…



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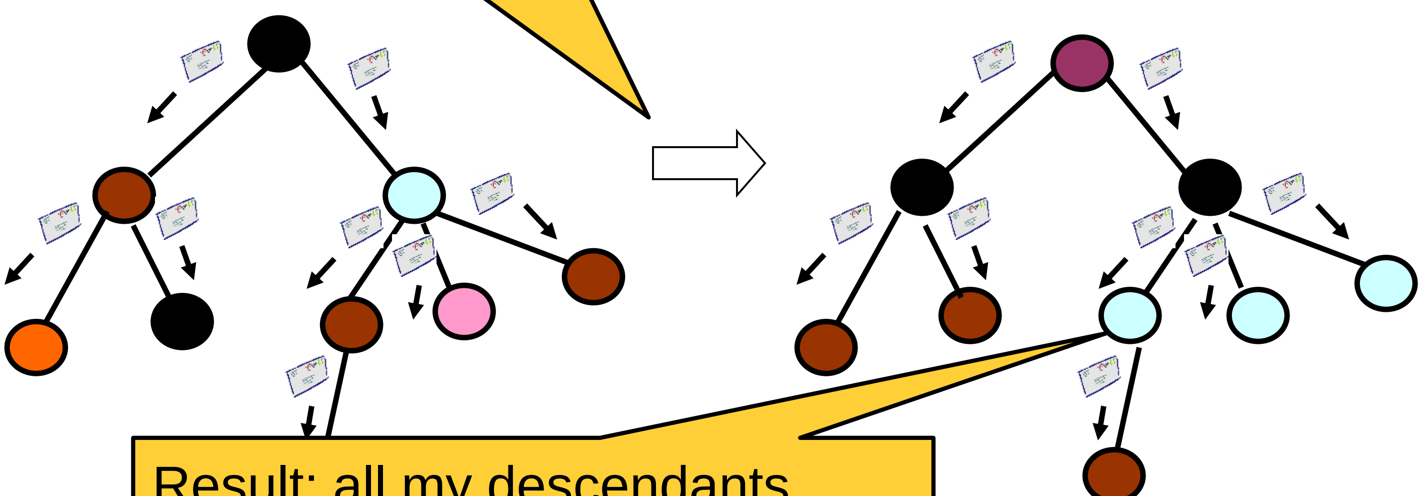

Result: all my descendants have same color! At most 2 colors are occupied: father and descendants! 3rd color free!

Observation: Shift Down

Shift Down

Each node v concurrently does:
recolor v with color of parent

Formally…

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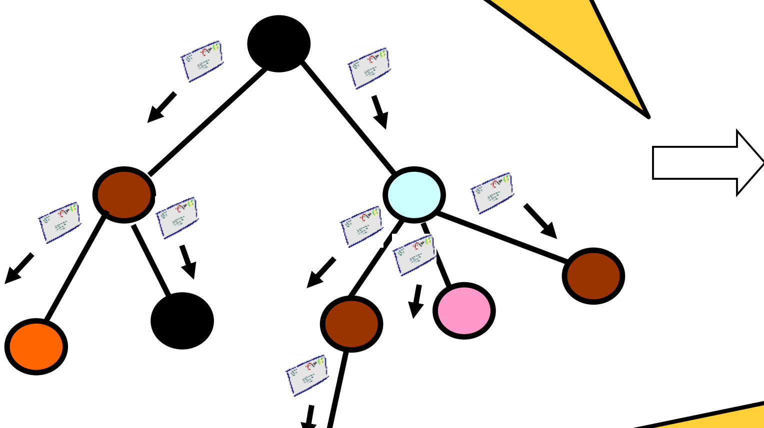

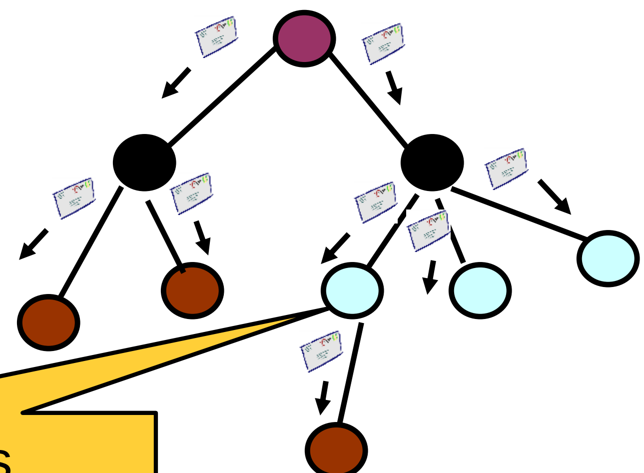

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6-to-3

Each other node v does (in parallel):

1. Run „**6-Colors**“ for log*(n) rounds
2. For x=5,4,3:
   1. Perform **Shift Down**
   2. If (c_v=x) choose new color c_v ϵ {0,1,2} according „**first free**“ principle

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Why still log* n time?

Just 3 more rounds!

6-to-3

Why not do in same step?

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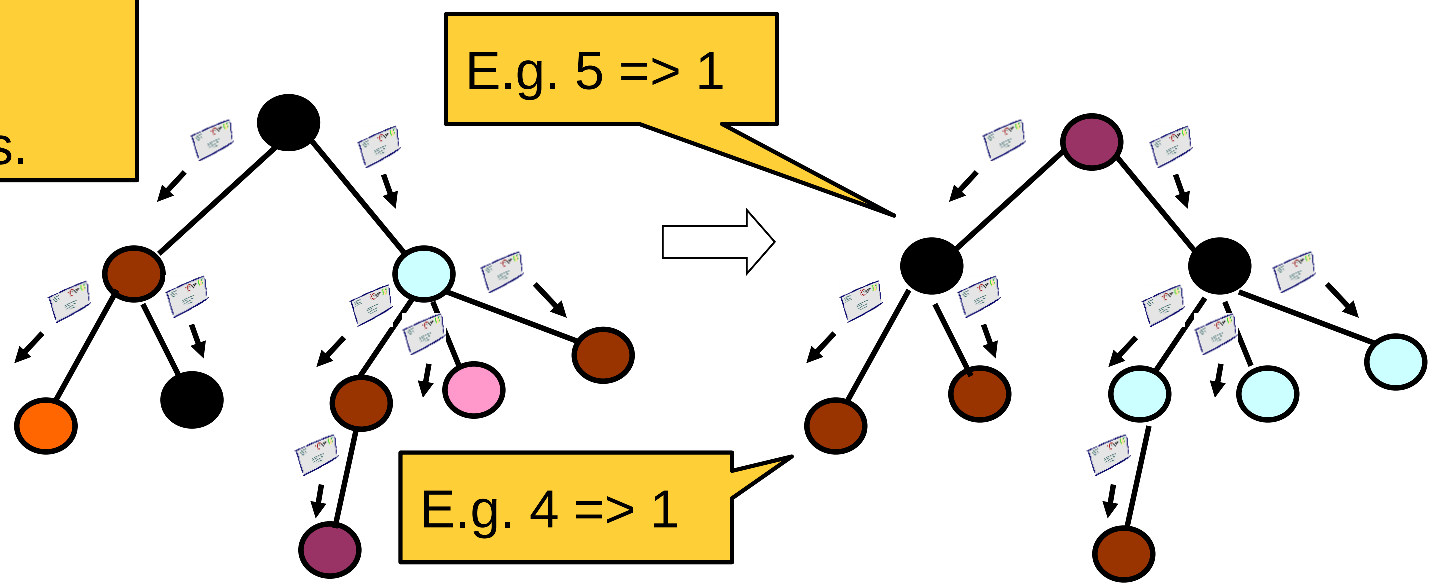
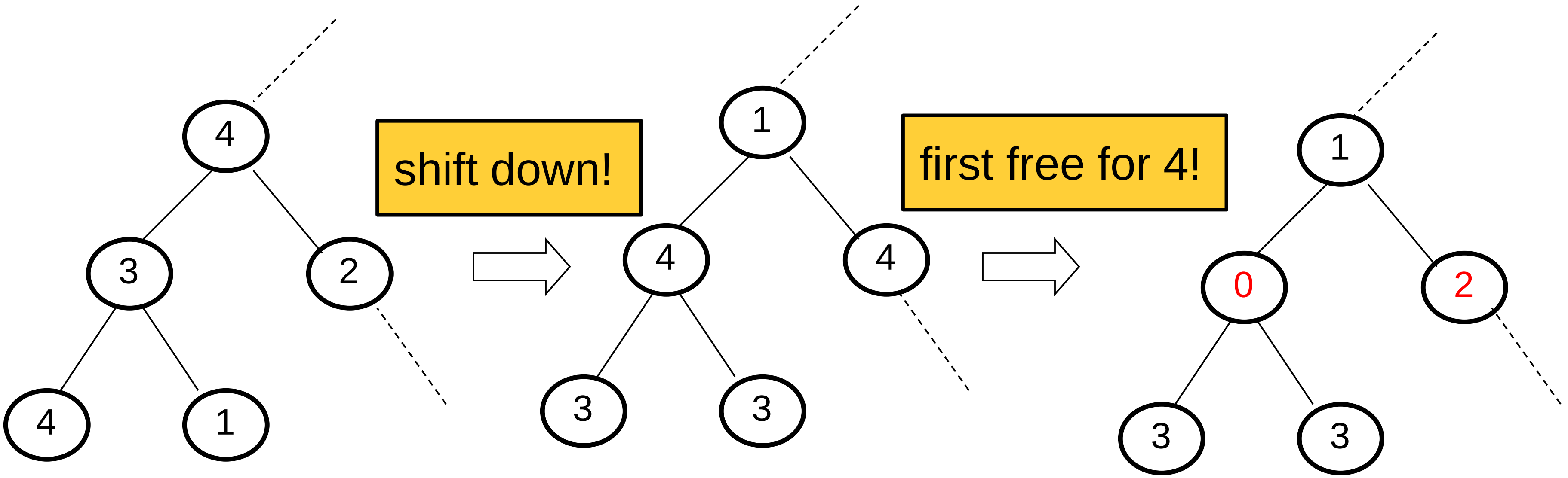
Could be harmful: same 3rd color! Need to do it for independent sets.

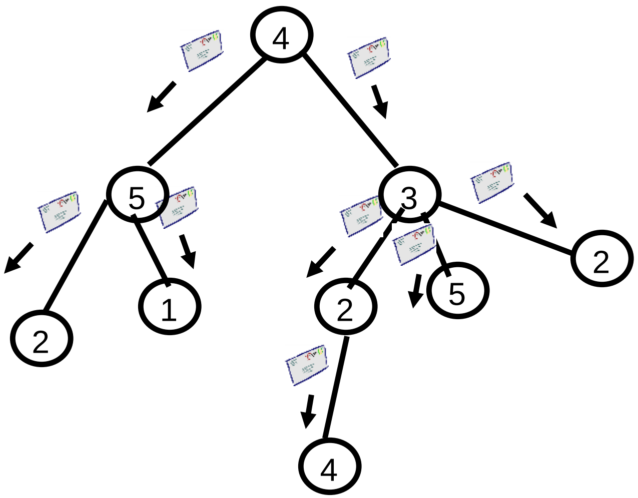


Example: Shift Down + Drop Color 4

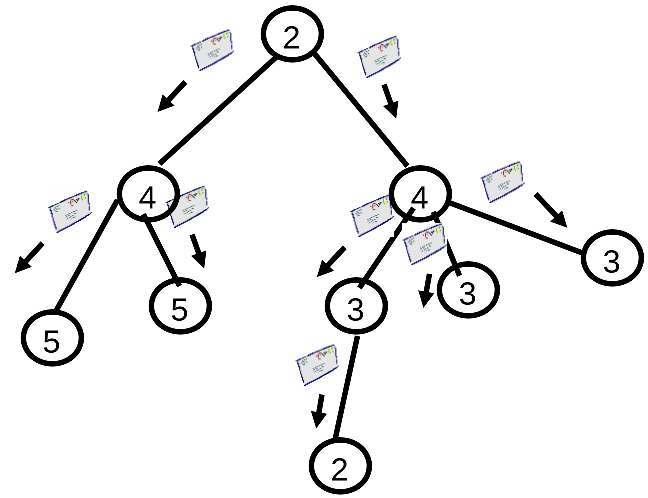


Siblings no longer have same color: must do shift down again first!

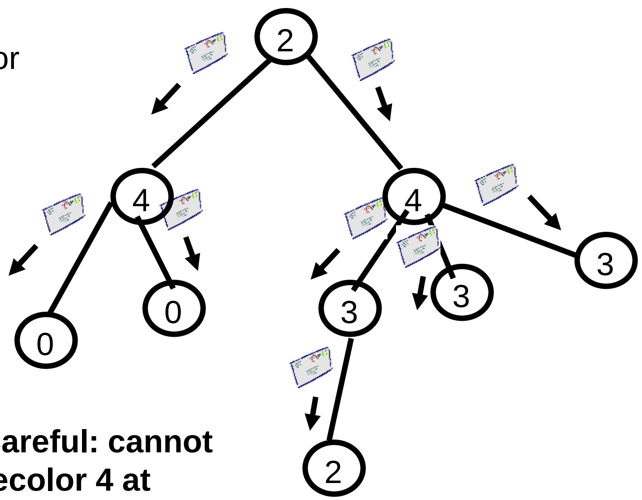
Example: 6-to-3



shift down

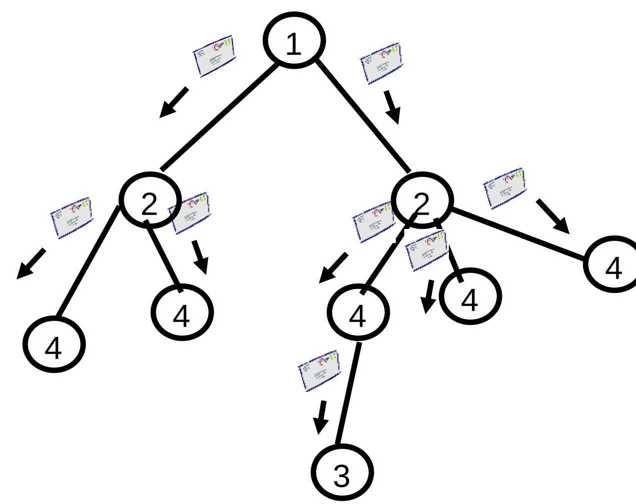


new color for 5: **first free**



Careful: cannot recolor 4 at same time!

shift down



Remark: Optimality

One can show that no local algorithm can 3-color a graph faster than in O(log* n).

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One can show that no local algorithm can 3-color a graph faster than in $O(\log^* n)$.

In fact:
in 0 rounds: $\geq$ n colors
in 1 round: $\geq$ log n colors
in 2 rounds: $\geq$ loglog n colors
etc.!

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in 0 rounds: ≥ n colors
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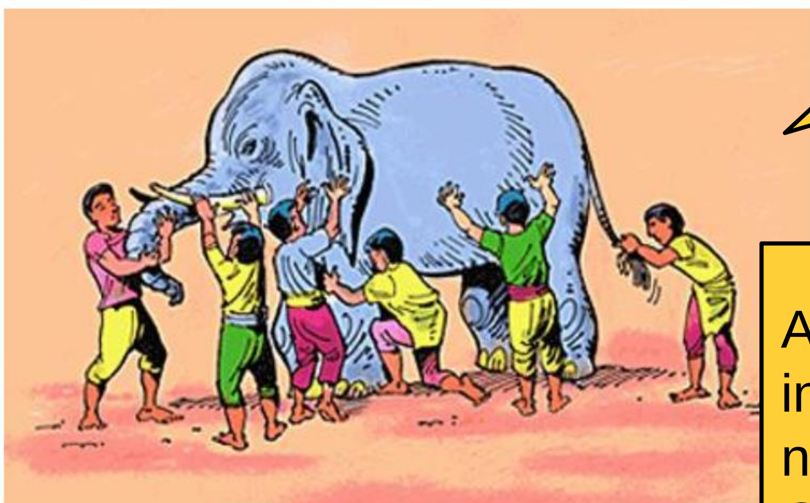
One can show that no local algorithm can 3-color a graph faster than in O(log* n).

Proof idea: Recall the elephant!

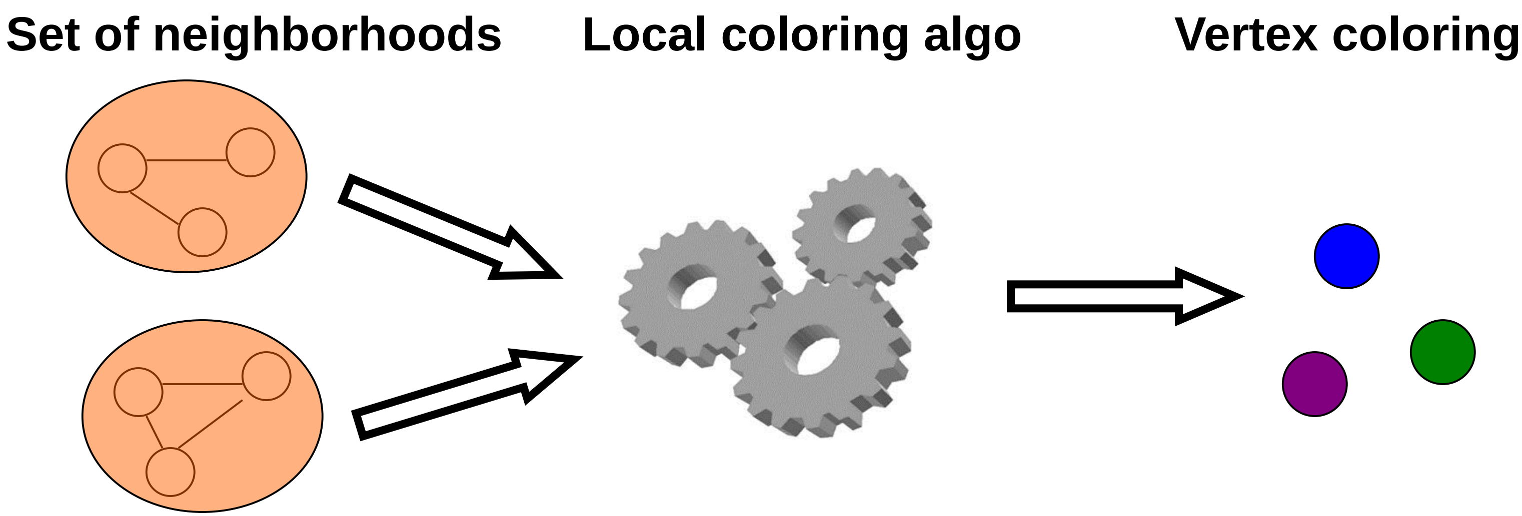
A local coloring algorithm can be seen as a function:

f: neighborhood → color

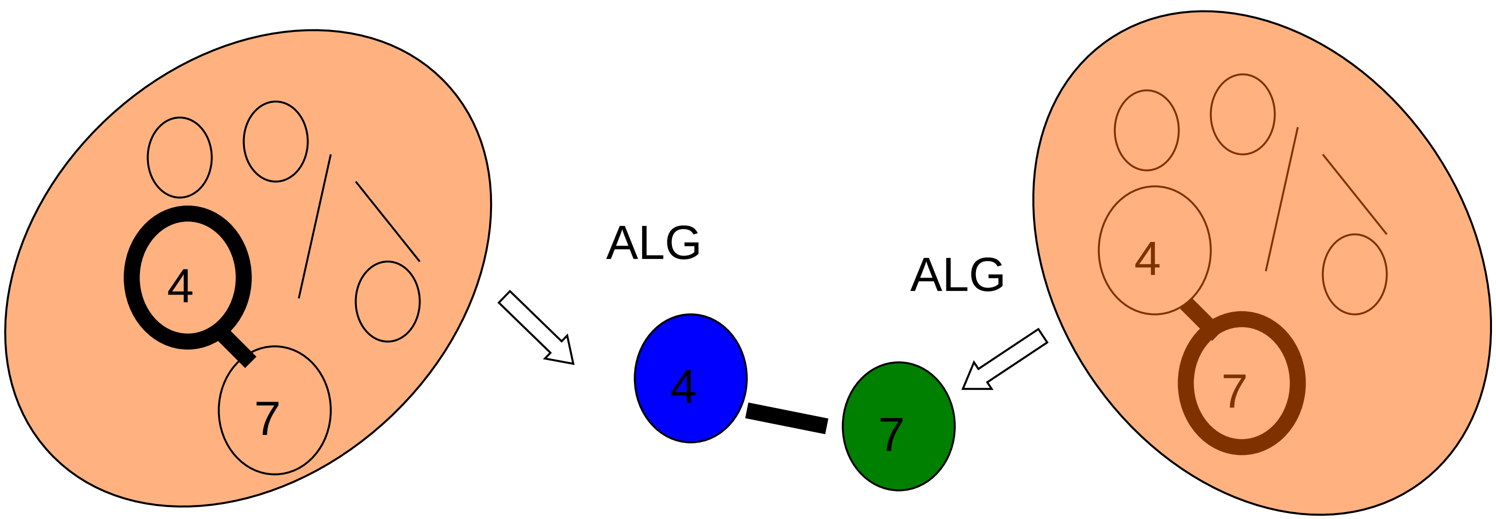


A deterministic algorithm needs to decide in the same way given same neighborhood: risk illegal coloring. Only with communication neighborhoods start look different and require less colors.

Lower Bound



Set of neighborhoods
Local coloring algo
Vertex coloring

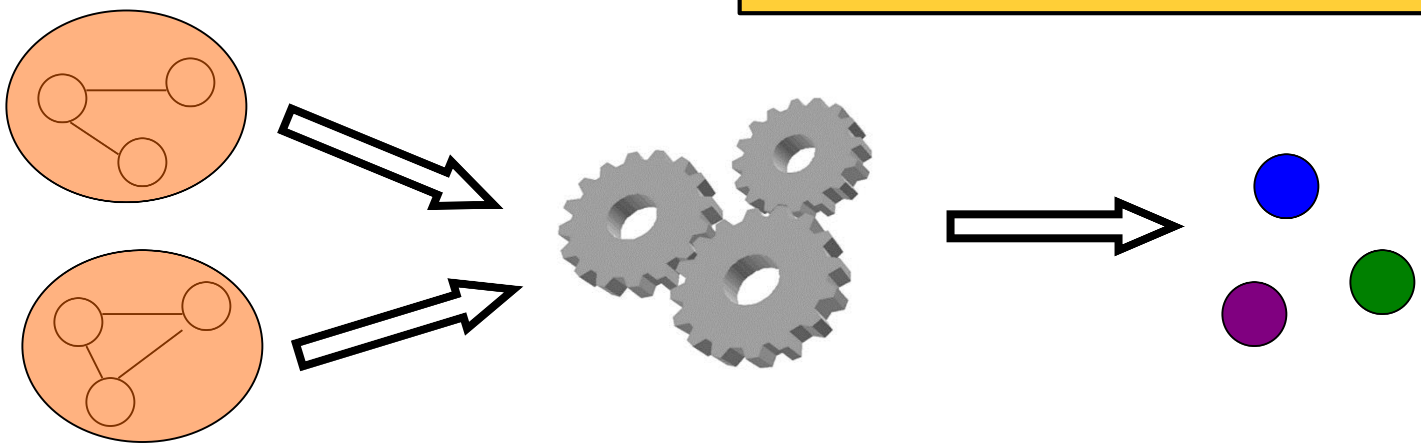


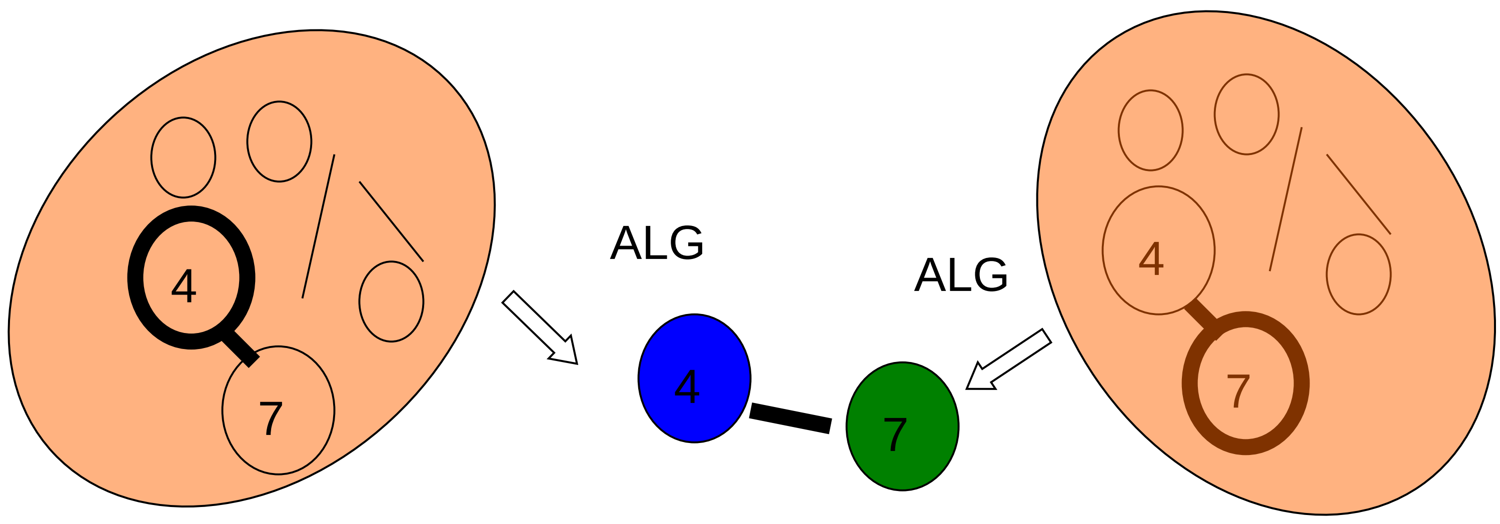
4
7
ALG
4
7
ALG
4
7

Lower Bound

Can reduce problem of finding lower bound to determine chromatic number of special neighborhood dependency graphs.

Set of neighborhoods **Local**





Concluding Remarks

Can we reduce to 2 colors?

Not without increasing runtime significantly!
(Linear time, more than exponentially worse!)

Simple on purpose: results more general!

log* runtime is also possible on more general graphs
Many results: see ACM PODC conference!

Where can I learn more?

- Distributed Computing book by David Peleg
- Lecture notes by Roger Wattenhofer ETH Zurich
- ACM Survey by Jukka Suomela
- Research: ACM PODC Conference