An Introduction to Combinatorial Optimization

Markos Epitropou

National Technical University of Athens

markosepitropou@gmail.com

July 1, 2014

Overview

Duality

Combinatorial Statement - Example

In a graph, the smallest number of edges in a path between two specified vertices s and t is equal to the maximum number of s - t cuts (i.e. subsets of edges whose removal disconnects s and t).

Primal

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & Ax = b \\ & x \geq 0 \end{aligned}$$

Dual

$$\begin{aligned} \max\ & b^T y \\ \text{s.t. } & A^T y \leq c \end{aligned}$$

Duality

$$
\begin{aligned}
\min\ & c_1^T x_1 + c_2^T x_2 + c_3^T x_3 \\
\text{s.t.}\ & A_{11}x_1 + A_{12}x_2 + A_{13}x_3 = b_1 \\
& A_{21}x_1 + A_{22}x_2 + A_{23}x_3 \geq b_2 \\
& A_{31}x_1 + A_{32}x_2 + A_{33}x_3 \leq b_3 \\
& x_1 \geq 0 \\
& x_2 \leq 0
\end{aligned}
$$

$$
\begin{aligned}
\max\ & b_1^T y_1 + b_2^T y_2 + b_3^T y_3 \\
\text{s.t.}\ & A_{11}^T y_1 + A_{21}^T y_2 + A_{31}^T y_3 \leq c_1 \\
& A_{12}^T y_1 + A_{22}^T y_2 + A_{32}^T y_3 \geq c_2 \\
& A_{13}^T y_1 + A_{23}^T y_2 + A_{33}^T y_3 = c_3 \\
& y_2 \geq 0 \\
& y_3 \leq 0
\end{aligned}
$$

Farkas' Lemma

Farkas' Lemma

Exactly one of the following is true for the system $Ax = b, x \geq 0$:

- There is x such that $Ax = b, x \geq 0$.
- There is y such that $A^T y \geq 0$ but $b^T y < 0$

Complementary Slackness

The duality gap $c^T x - b^T y$ is a measure of optimality.

Complementary Slackness

Let x^*, (y^*, s^*) be feasible for (P), (D) respectively. The following are equivalent:

1. x^* is an optimal solution to (P) and (y^*, s^*) is an optimal solution to (D).
2. $(s^*)^T x^* = 0$
3. $x_j^* s_j^* = 0$, $\forall j = 1, \ldots, n$
4. if $s_j^* > 0$ then $x_j^* = 0$

Totally Unimodular Matrices

Totally Unimodular Matrix

A matrix A is totally unimodular if every square submatrix has determinant 0, $+1$, or -1. In particular, this implies that all entries are 0, $+1$, or -1.

Integer Vertices

If A is totally unimodular and b is an integer vector, then $P = \{x : Ax \leq b\}$ has integer vertices.

Primal - Dual Methods

Starting with a dual feasible solution y, the method searches for a primal feasible solution x satisfying the complementary slackness condition with respect to y. If such a primal feasible solution x is found, x and y form a pair of optimum solutions. If no such primal solution is found, the method prescribes a modification of y, after which the method iterates.

- Solve $x' \geq 0, A'x' = b$ (Complementary slackness)
- If no x' exists, there exists y' such that ${y'}^T A \leq 0$ and ${y'}^T b > 0$
- Let α the largest real number satisfying $(y_0 + \alpha y')^T A \leq c^T$.

Polyhedral Combinatorics

- Max - Min Relations
- Defining a polytope with linear inequalities
- Seperation Oracle

Bipartite Matching

König's Theorem

For any bipartite graph $G = (V, E)$, the cardinality of the maximum matching $max\_match(G)$ and the cardinality of the minimum vertex cover $min\_vc(G)$ are the same.

Bipartite Matching

For all bipartite graphs, the incidence matrix A is totally unimodular.

$$\max\{1^T x : x \in P_{match(G)}\} = \max\{1^T x : x \geq 0, \forall v \in V \sum_{e \in \delta(v)} x_e \leq 1\} =$$

$$\max\{1^T x : x \geq 0, Ax \leq 1\} = \min\{1^T x : y \geq 0, A^T y \geq 1\} =$$

$$\min\{1^T y : y \geq 0, \forall (u,v) \in E y_u + y_v \geq 1\}$$

Matching Polytope

Matching Polytope in Bipartite Graphs

If G is bipartite, the matching polytope is determined by

- $x_e \geq 0$, for each edge e
- $x(\delta(v)) \leq 1$, for each vertex v

Matching Polytope

For any graph $G = (V, E)$, the matching polytope is determined by:

- $x_e \geq 0$, for each edge e
- $x(\delta(v)) \leq 1$, for each vertex v
- $x(E[U]) \leq \lfloor \frac{1}{2}|U| \rfloor$, for each $U \subseteq V$ with $|U|$ odd

Non-bipartite Cardinality Matching

Tutte-Berge Formula

For any graph $G = (V, E)$, we have

$$\max_M |M| = \min_{U \subseteq V} \frac{1}{2}(|V| + |U| - o(G \setminus U))$$

where $o(G \setminus U)$ is the number of connected components of odd size of $G \setminus U$.

Optimality Conditions

A matching M is maximum if and only if there are no augmenting paths with respect to M.

How can we find M-augmenting paths or decide that none exist?

Edmond's Algorithm

Problem Formulation

b-flow

Given a digraph G, capacities $u : E(G) \to \mathcal{R}_+$, and numbers $b : V(G) \to \mathcal{R}$ with $\sum_{v \in V(G)} b(v) = 0$, a **b-flow** in (G, u) is a function $f : E(G) \to \mathcal{R}_+$ with $f(e) \leq u(e)$ for all $e \in E(G)$ and $\sum_{e \in \delta^+(v)} f(e) - \sum_{e \in \delta^-(v)} f(e) = b(v)$ for all $v \in V(G)$.

- Input: A digraph G, capacities $u : E(G) \to \mathcal{R}_+$, numbers $b : V(G) \to \mathcal{R}$ with $\sum_{v \in V(G)} b(v) = 0$, and weights $c : E(G) \to \mathcal{R}$.
- Output: A b-flow f whose cost $c(f) = \sum_{e \in E(G)} f(e)c(e)$ is minimum (if one exists).

Problem Formulation

$$\min \sum_{e \in E(G)} c(e)x_e$$

$$\text{s.t.} \sum_{e \in \delta^+(v)} x_e - \sum_{e \in \delta^-(v)} x_e = b(v) (v \in V(G))$$

$$x_e \leq u(e) \qquad (e \in E(G))$$

$$x_e \geq 0 \qquad (e \in E(G))$$

- Circulation: A flow satisfying $\sum_{e \in \delta^+(v)} f(e) - \sum_{e \in \delta^-(v)} f(e) = 0$ for all $v \in V(G)$.

Flow Decomposition Theorem

Flow Decomposition Theorem

Let (G, u, s, t) be a network and let f be an $s-t$-flow in G. Then there exists a family $\mathcal{P}$ of $s-t$-paths and a family $\mathcal{C}$ of circuits in G along with weights $h : \mathcal{P} \cup \mathcal{C} \Rightarrow \mathcal{R}_+$ such that $f(e) = \sum_{P \in \mathcal{P} \cup \mathcal{C} : e \in E(P)} h(P)$ for all $e \in E(G)$ and $|\mathcal{P}| + |\mathcal{C}| \leq |E(G)|$.

Optimality Conditions

Optimality Conditions

Let (G, u, b, c) be an instance of the MINIMUM COST FLOW PROBLEM. A b-flow f is of minimum cost if and only if it satisfies the negative cycle optimality conditions: namely the residual network G_f contains no negative cost (directed) cycle.

Minimum Mean Cycle-Cancelling Algorithm

Algorithm:

1. Find a b-flow f.
2. Find a circuit C in G_f whose mean weight is minimum. If C has nonnegative total weight (or G_f is acyclic) **then stop**.
3. Compute $\gamma = \min_{e \in E(C)} u_f(e)$. Augment f along C by γ. **Go to step** 2.

How do you implement 1 **and** 2?