# Two path coloring problems with applications in optical networking

1. The Routing and Path Multi-Coloring Problem
2. The Maximum Routing and Path Coloring Problem

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NTUA – $\mu\Pi\lambda\forall$

## Graph-theoretic representation

| | | |
|---|---|---|
| network | $\longleftrightarrow$ | graph |
| requests | $\longleftrightarrow$ | pairs of nodes |
| connections | $\longleftrightarrow$ | paths |
| wavelengths | $\longleftrightarrow$ | colors |

*Arising graph problems are interesting per se.*

## Routing and Path Coloring Problems: Other Applications

### Color interpretation

Generic communication networks: time-slots.

Vehicle routing: tracks or departure time slots.

Scheduling: processors.

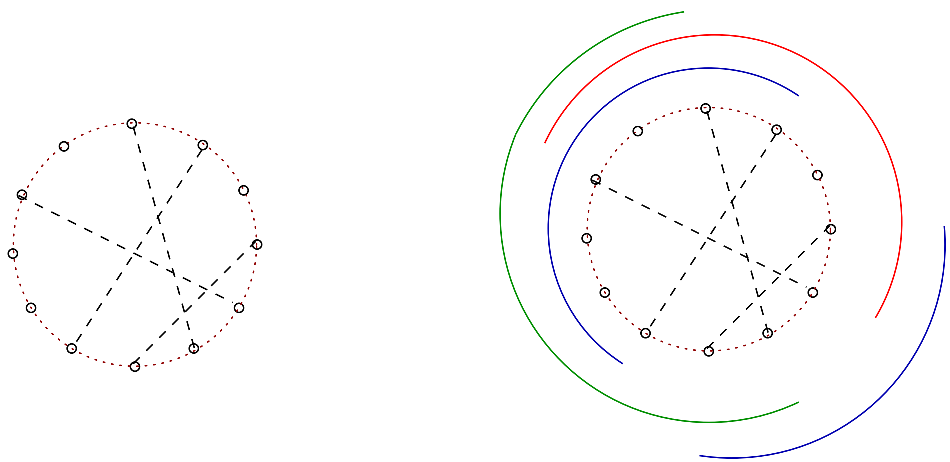
Compiler optimization: memory locations.

## Routing and Path Coloring – RPC (aka RWA)

*Instance:* Graph $G(V, E)$, collection of requests (pairs of nodes) $R = \{(s_1, t_1) \ldots (s_m, t_m)\}$.

*Feasible Solution:* path and color assignment to requests in $R$ such that edge-intersecting paths have different colors.

*Goal:* minimize number of colors.



# Known results for RPC

- *Chains (interval coloring):* poly-time solvable [folklore, e.g. Tucker '75].
- *Rings:* NP-hard, approximation ratio 2 [Raghavan, Upfal '93], [Kaklamanis, Mihail, Rao '95] (directed). *Pre-routed version (circular-arc coloring)*: approx. ratio $3/2$ [Karapetian '80].
  Improved to $\approx 1.68$ and $\approx 1.37$ (for heavily loaded instances) respectively, using randomized algorithms [Kumar '98].
- *Trees:* NP-hard, approximation ratio 1.1 (asympt.) by 1.1 edge-coloring [Nishizeki, Kashiwagi '90]. DIRECTED: $5/3$-approx. [Erlebach, Jansen, Kaklamanis, Mihail, Persiano '99].
- *Meshes:* approx. ratio $\log\log n$ [Rabani '96]. *Pre-routed version*: No better approx. ratio than GRAPH COLORING [Nomikos '96], i.e. non-approximable within $|R|^{1/7-\epsilon}$ [Bellare, Goldreich, Sudan '98].

## Possible approaches - variations

- directed / undirected paths
- pre-routed or not
- minimization / maximization (weighted or not)
- full / limited / no wavelength conversion
- on-line / off-line
- multiple / single fiber(s)
  - full / limited switching.

# The Routing and Path Multi-Coloring Problem

## Network Model

- Undirected (full-duplex).
- No wavelength conversion.
- Fixed number of wavelengths.
- Multiple fibers, full switching.
- Only active fibers are charged.

*Goal:* minimize (the cost of) active fibers.

*Related work*: [Li, Simha '00], [Simon, Margara '00]. Fixed number of fibers everywhere, minimization of wavelengths per fiber.

## Routing and Path Multi-Coloring (RPMC)

*Instance:* graph $G(V, E)$, collection of requests (pairs of nodes), number of colors $w$.

*Feasible Solution:* Path and color assignment to $R$ (collisions allowed)

*Goal:* minimize

$$\sum_{e \in E} \mu(e)$$

$\mu(e)$: the maximum multiplicity of any color on edge $e$

number of fibers per link $\longleftrightarrow$ multiplicity of colors per edge

# Complexity and approximability of RPMC

***Chains:*** poly-time solvable [Nomikos, P., Zachos, *IPL* 2001]
Improved complexity: [Winkler, Zhang, SODA'03]

***Rings, Stars, Spiders:*** NP-hard, locally near-optimal solutions,

$$SOL \leq OPT + |E| \leq 2OPT$$

[Nomikos, P., Zachos, *IPL* 2001] (pre-routed rings, rings, stars)
[Nomikos, P., Potika, Zachos, PCI'01] (spiders)
[Nomikos, P., Potika, Zachos, Networking'04] (non-uniform costs, directed versions)

***Trees:*** NP-hard, approximation ratio 4 [Chekuri, Mydlarz, Shepherd, ICALP'03] and $1+4|E|\log|V|/OPT$ [Erlebach, P., Potika, Stefanakos, WG'03]

*Other multiple-fiber problems*: [Li, Simha, INFOCOM'00], [Simon, Margara, ICALP'00], [Erlebach, P., Potika, Stefanakos, WG'03].

## Results in this presentation

***Chains:*** poly-time solvable.

[Nomikos, P., Zachos '97]

***Rings, Stars, Suns:*** NP-hard, locally near-optimal solutions,

$$SOL \leq OPT + |E| \leq 2OPT$$

[Nomikos, P., Zachos '97] (pre-routed rings),

[Nomikos, P., Zachos '00] (rings, stars)

[Nomikos, P., Potika, Zachos '01] (spiders or suns)

[Nomikos, P., Potika, Zachos '04] (costs on edges, directed versions)

# Notation and Lower Bounds

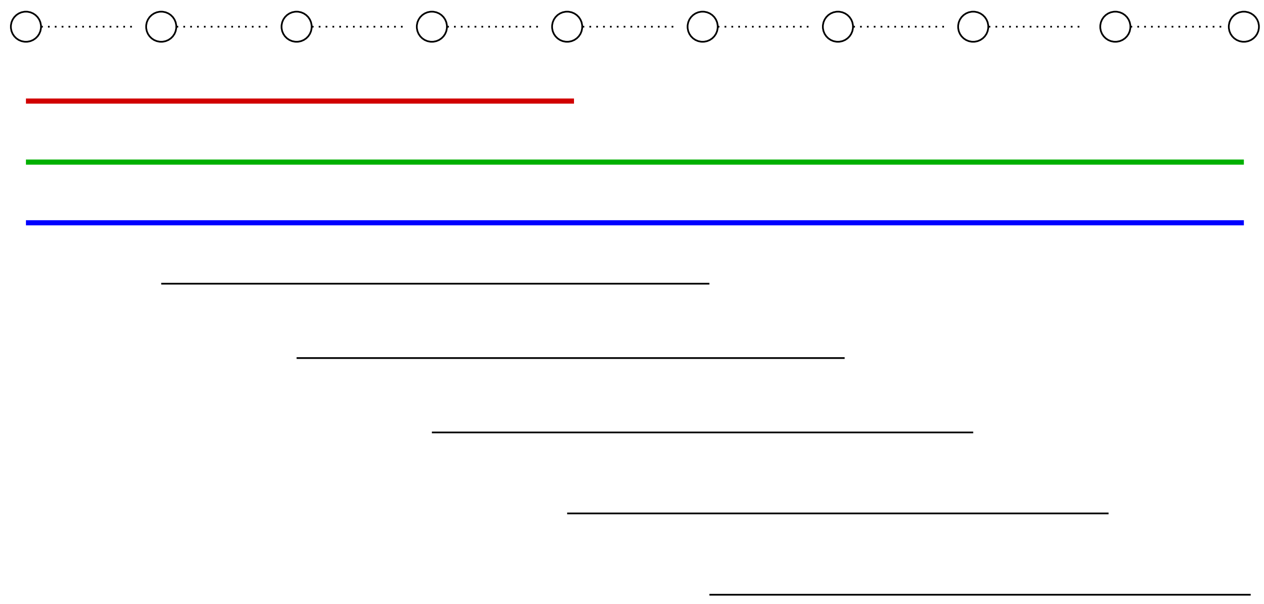
- $n$ = number of nodes
- $m$ = number of requests or connections
- $w$ = number of colors (wavelengths)
- $L(e)$ = load of edge $e$
- $L$ = maximum load

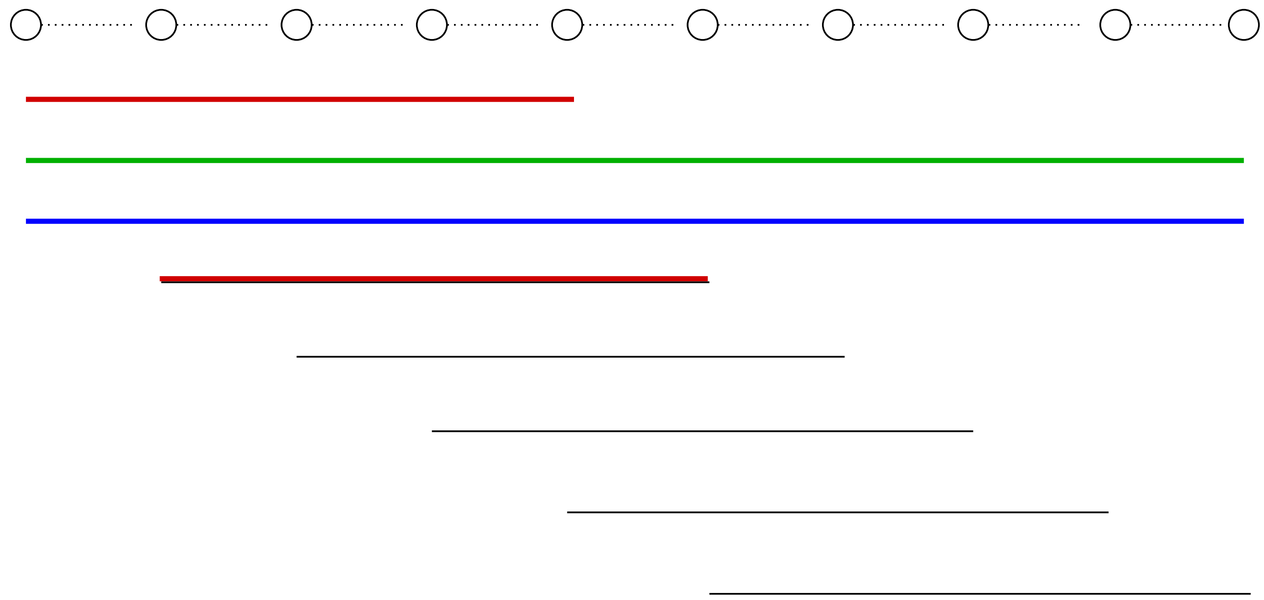
**Lower bound:** $\mu(e) \geq \lceil \frac{L(e)}{w} \rceil$, hence:

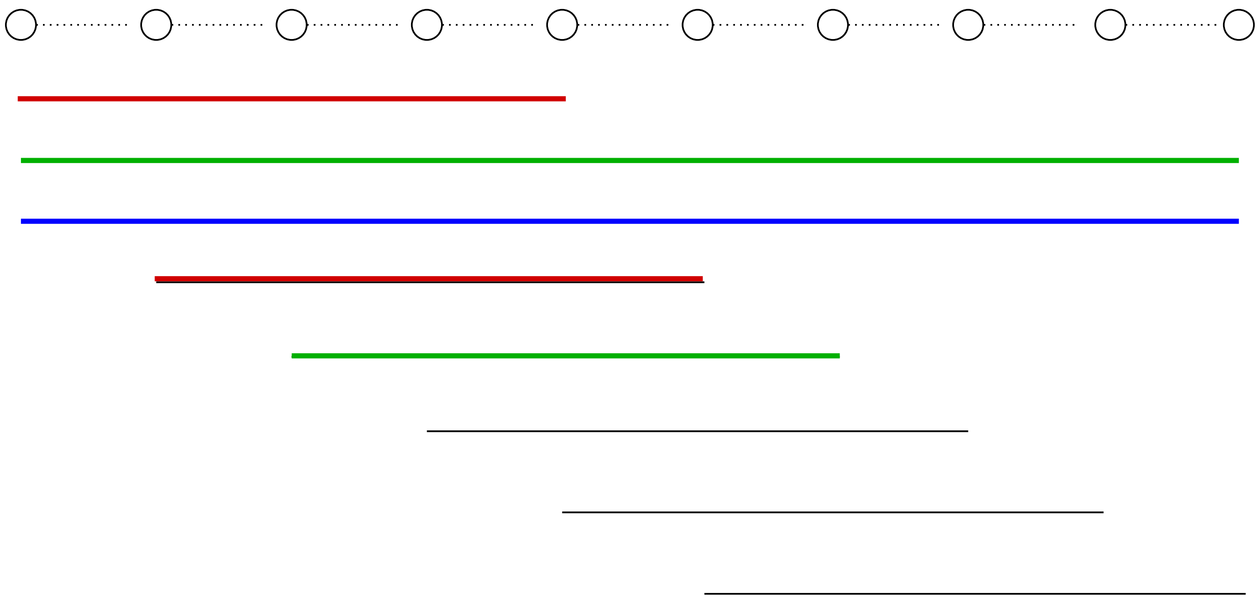
$$OPT \geq \sum_{e \in E} \lceil \frac{L(e)}{w} \rceil$$

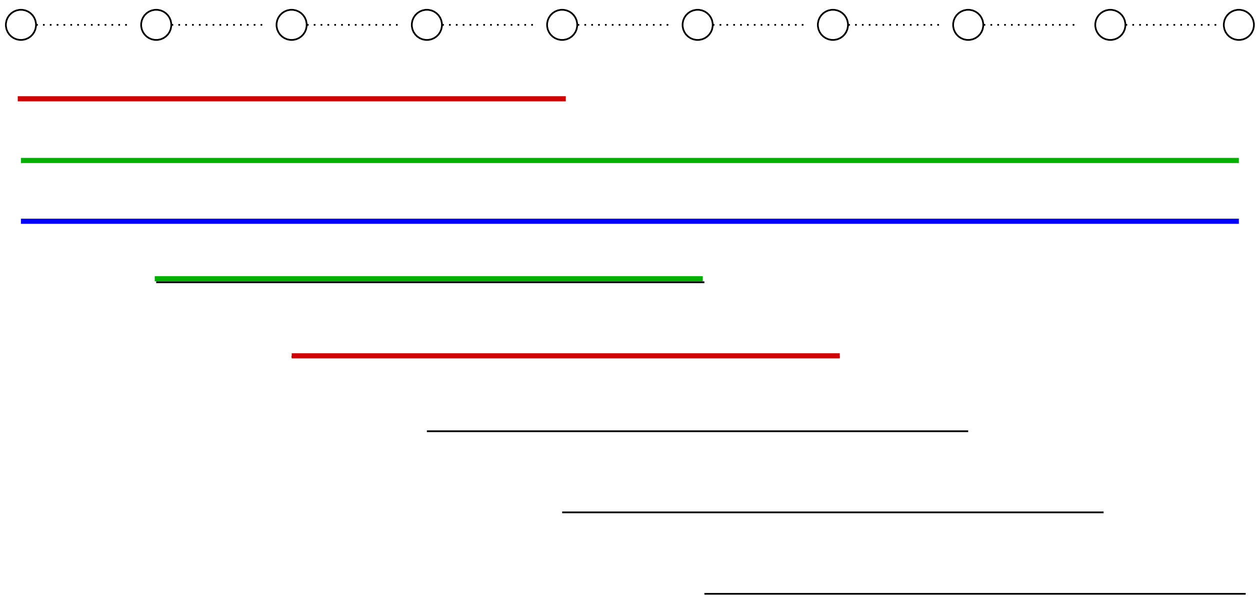
## RPMC in Chains

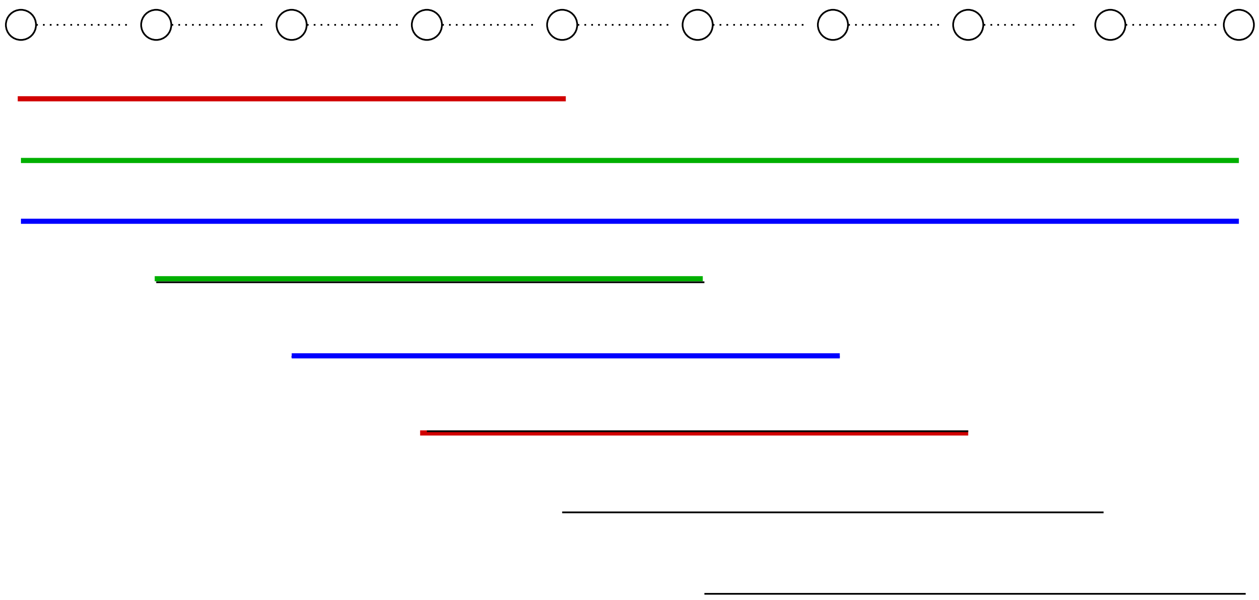
*Difficulty:* no greedy technique seems adequate.

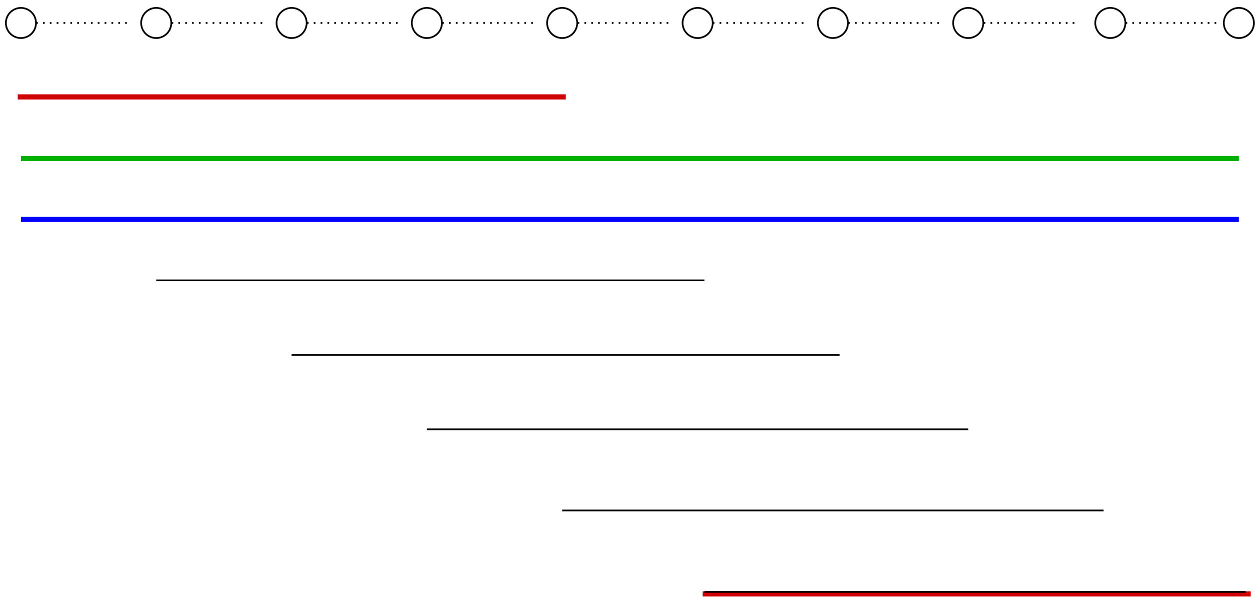


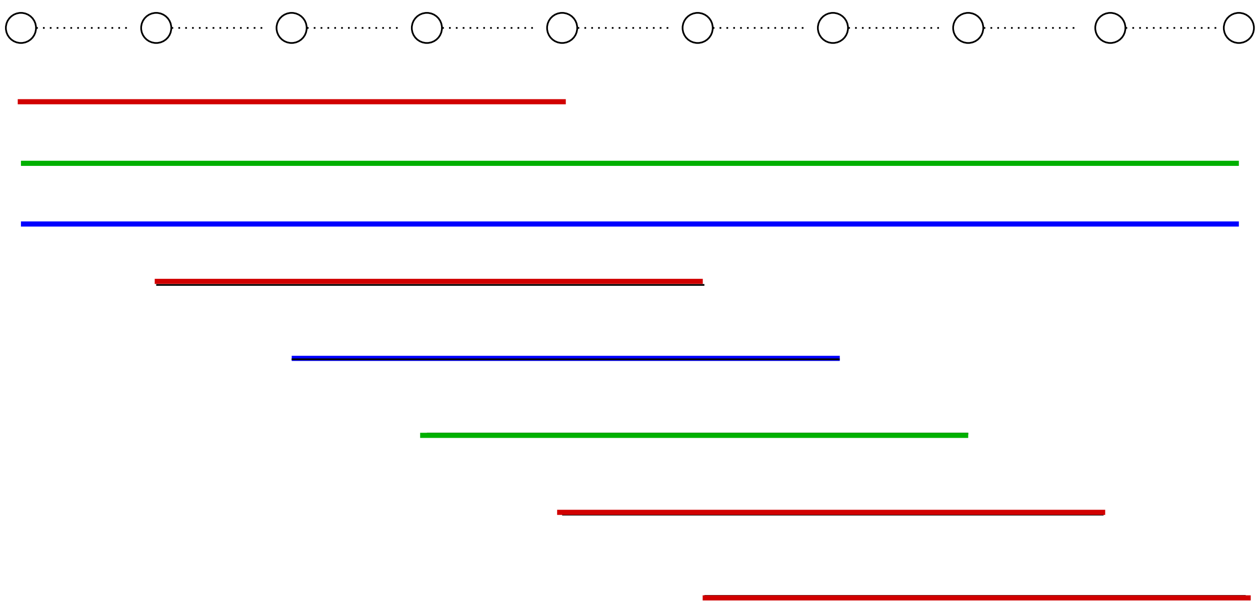












## Technique: reduction to Bipartite Edge Coloring

### The Algorithm

- Fill to complete multiples of $w$ and join.
- Gather starts and ends into groups of $w$, creating a bipartite graph.
- Apply bipartite edge coloring.

*Complexity:* $O(m' \log m'),\ m' \leq m + nw.$

## RPMC in Stars, Suns and Rings: NP-hardness

Decision versions of multi-coloring problems are generalizations of the decision versions of the corresponding (single) coloring problems.

Single-fiber problems for stars, spiders and rings are NP-hard.

# Algorithm for Stars

- Assign arbitrary direction to paths.
- On each 'ray', partition separately inward and outward paths to groups of $w$ paths.
- Construct bipartite graph of degree $w$.
- Edge-color the bipartite graph.

*Proof of correctness:* based on the fact $\lceil a \rceil + \lceil b \rceil \leq \lceil a + b \rceil + 1$.

$$SOL(e) = \lceil \frac{L_i(e)}{w} \rceil + \lceil \frac{L_o(e)}{w} \rceil \leq \lceil \frac{(L_i(e) + L_o(e))}{w} \rceil + 1 \leq OPT(e) + 1$$

$$SOL \leq OPT + |E|$$

*Paths of length 1 can be directed in a more clever way.*

## Algorithm for Spiders

- Assign arbitrary direction to paths that traverse the center. Assign inward direction to the remaining paths.
- On each 'ray', partition independently the inward and the outward paths to groups of $w$ paths. For inward paths, first apply the chain technique (filling to multiples of $w$ and joining). Collect outward paths starting from the end.
- Construct bipartite graph of degree $w$.
- Edge-color the bipartite graph.

*Correctness:* Argument similar to the one for stars.

*With appropriate modification the algorithm gives optimal solution for the directed problem. [Nomikos, P., Potika, Zachos '04]*

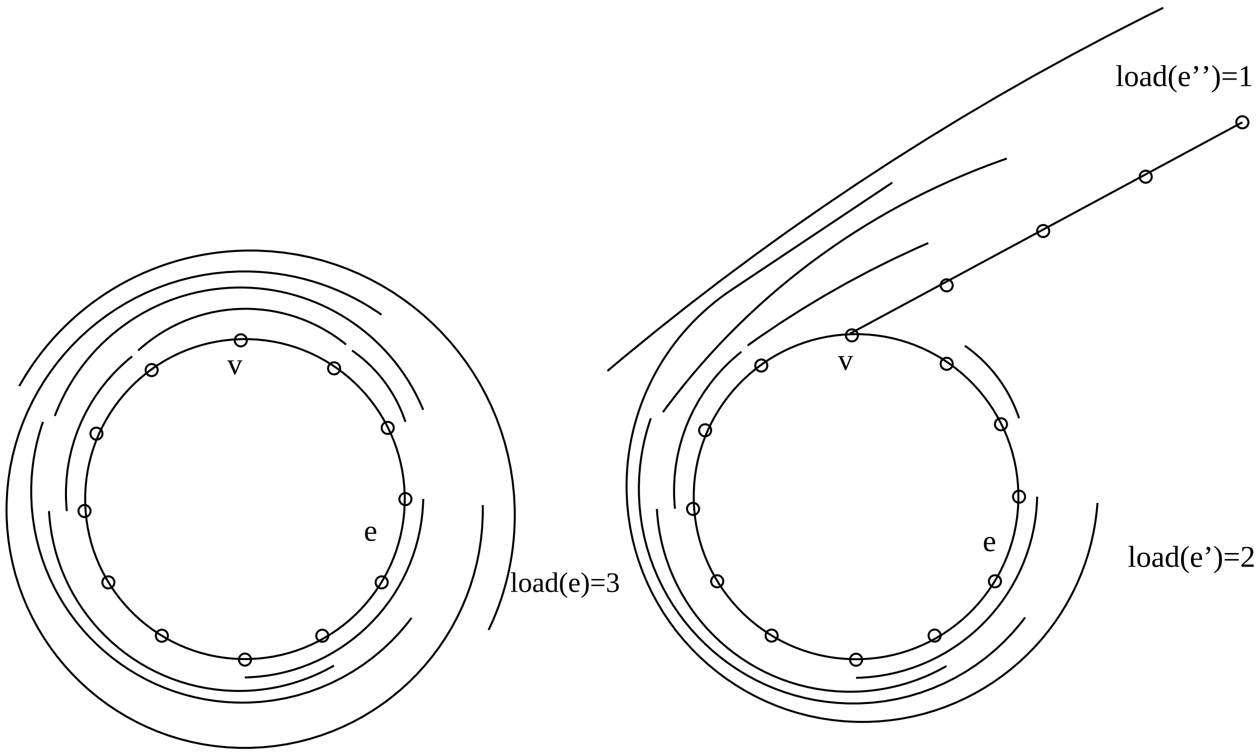
## Pre-routed Rings Algorithm: RING-PMC Problem

- Find a 'good' break-point.
- Transform to chain instance.
- Apply chain algorithm.

*Correctness:* Argument similar to the one for stars.

$$SOL \leq OPT + s \leq OPT + |E| - 2$$

# Ring Unfolding Technique



# Algorithm for Rings (RPMC)

- Perform shortest-path routing.
- Apply the Ring-PMC algorithm.

*Proof of correctness:* Let $e'$ be the edge 'opposite' to $e$:

$$L(e) + L(e') \leq L^*(e) + L^*(e')$$

Additive error due to routing: $\lfloor |E|/2 \rfloor$ ($a + b \leq c + d \Rightarrow \lceil a \rceil + \lceil b \rceil \leq \lceil c \rceil + \lceil d \rceil + 1$) .

Additive error due to Ring-PMC algorithm: $s \leq \lfloor |E|/2 \rfloor - 1$.

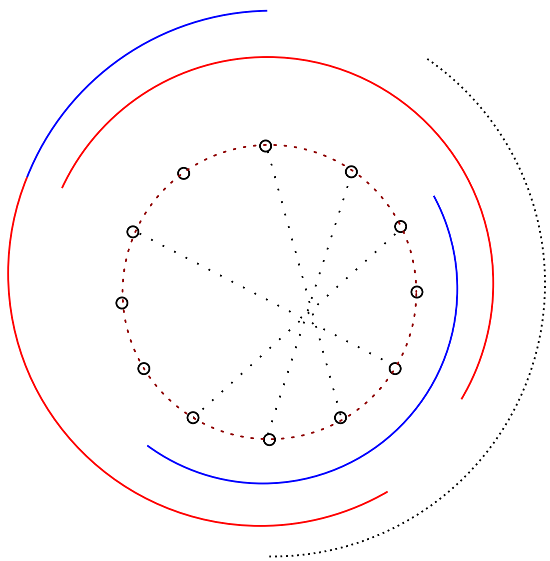
Overall approximation: $SOL \leq OPT + |E| - 1$.

## Summary

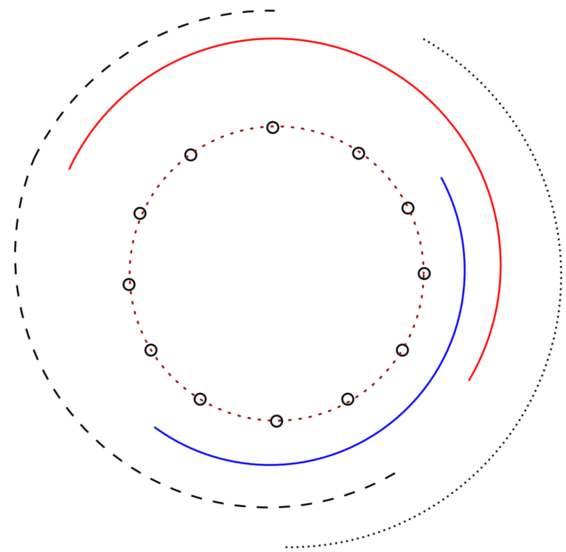
- Optimal solution for chains.
- Locally near-optimal solution for stars, suns and rings.
- All results extend to graphs with edge costs [NPPZ'04].
- The technique for suns yields an exact algorithm for the directed problem [NPPZ'04].
- $OPT + 1$ approximation algorithm for the problem of minimizing the maximum number of fibers.

# The Maximum Routing and Path Coloring Problem

# Examples of Request Blocking



*With*
wavelength conversion



*Without*
wavelength conversion

## Network Model

- Dual fiber: each link consists of two opposite-directed fibers.
  - Full-duplex communication: can be represented by undirected graph and requests.
  - One-way communication: can be represented by bidirected graph and directed requests.
- Requests may be pre-routed or not.
- No wavelength conversion.
- Number of wavelengths is part of the input.

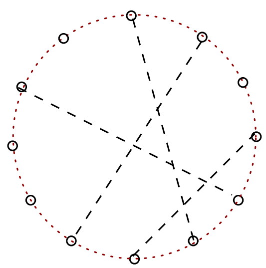
*Goal:* satisfy a maximum number of requests.

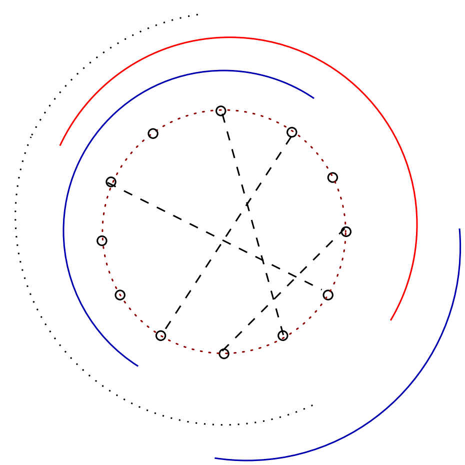
# Maximum Routing-and-Path-Coloring (MaxRPC)

*Instance:* graph $G(V,E)$, collection of requests (pairs of nodes) $R = \{(s_1,t_1)\ldots(s_m,t_m)\}$, number of colors $w$.

*Feasible Solution:* path and color assignment to a subset $A \subseteq R$ with $w$ colors and without color collisions.

*Goal:* maximize $|A|$.





## Variations

- DIRECTED MAXRPC: bidirected (symmetric directed) graph, directed requests and paths, collisions refer to arc-intersecting paths.
- MAXPC and DIRECTED MAXPC: routing is given (pre-routed requests).

# Complexity – approximability of MaxRPC

- *Chains:* poly-time solvable [Carlisle and Lloyd, *DAM* 1995].
- *Rings:* NP-hard, approximation ratio $(e-1)/e \approx 0.632$ [Wan and Liu, DIMACS'98].
  – MaxPC (Pre-routed version): approx. ratio $2/3 \approx 0.666$ [Nomikos and Zachos, ICALP-AlAsCo'97]; the same ratio for Directed MaxPC. Improved complexity: [Nomikos, P., Zachos, *Computer Networks* 2003].
- *Trees:* NP-hard, approximation ratio $\approx 0.632$ [Wan, Liu, DIMACS'98] using [Garg, Vazirani, Yannakakis, *Algorithmica* 1997].
  – Directed MaxRPC: [Erlebach, Jansen, ISAAC'98] for bounded-degree approx. ratio (e-1)/e; for general trees 0.451 ratio.
- *Meshes:* randomized constant approximation [Wan and Liu, DIMACS'98] using [Kleinberg and Tardos, STOC'95].
  – MaxPC (Pre-routed version): No approx. ratio better than $1/|R|^{\delta}$ [Nomikos and Zachos, ICALP-AlAsCo'97].

## Results in this presentation

- $(2/3)$-approximation algorithm for MaxRPC in rings based on the *Chain-and-Matching* technique. (Previous best ratio: $\approx 0.632$)
- A polynomial-time algorithm that solves Directed MaxRPC in rings optimally for one available color; this leads to A $0.632$-approximation algorithm for arbitrary number of colors.
- Introduction of a new matching problem: w-Bounded 2-Color Matching, and a $3/4$-approximation algorithm for this problem.
- A $(7/11)$-approximation algorithm for Directed MaxRPC in rings, based on the Chain-and-Matching technique, as well as on the algorithm for w-Bounded 2-Color Matching. The second algorithm slightly better ($7/11 \approx 0.636$).
- Improvement: $7/8$ for w-Bounded 2-Color Matching, yielding $15/23 \approx 0.652$ for Directed MaxRPC in rings. Also, exact randomized algorithm leading to randomized $2/3$-apx. algorithm.

# Notation

- $n$ = number of nodes
- $m$ = number of requests
- $w$ = number of colors (wavelengths)
- $L(e)$ = load of edge $e$
- $L$ = maximum load

## MaxRPC in Chains

- A simple algorithm: proceed from 'left' to 'right'. Wherever there are more paths than available colors, eliminate the longest paths.
- If $w > L$ it is possible to use only $L$ colors.

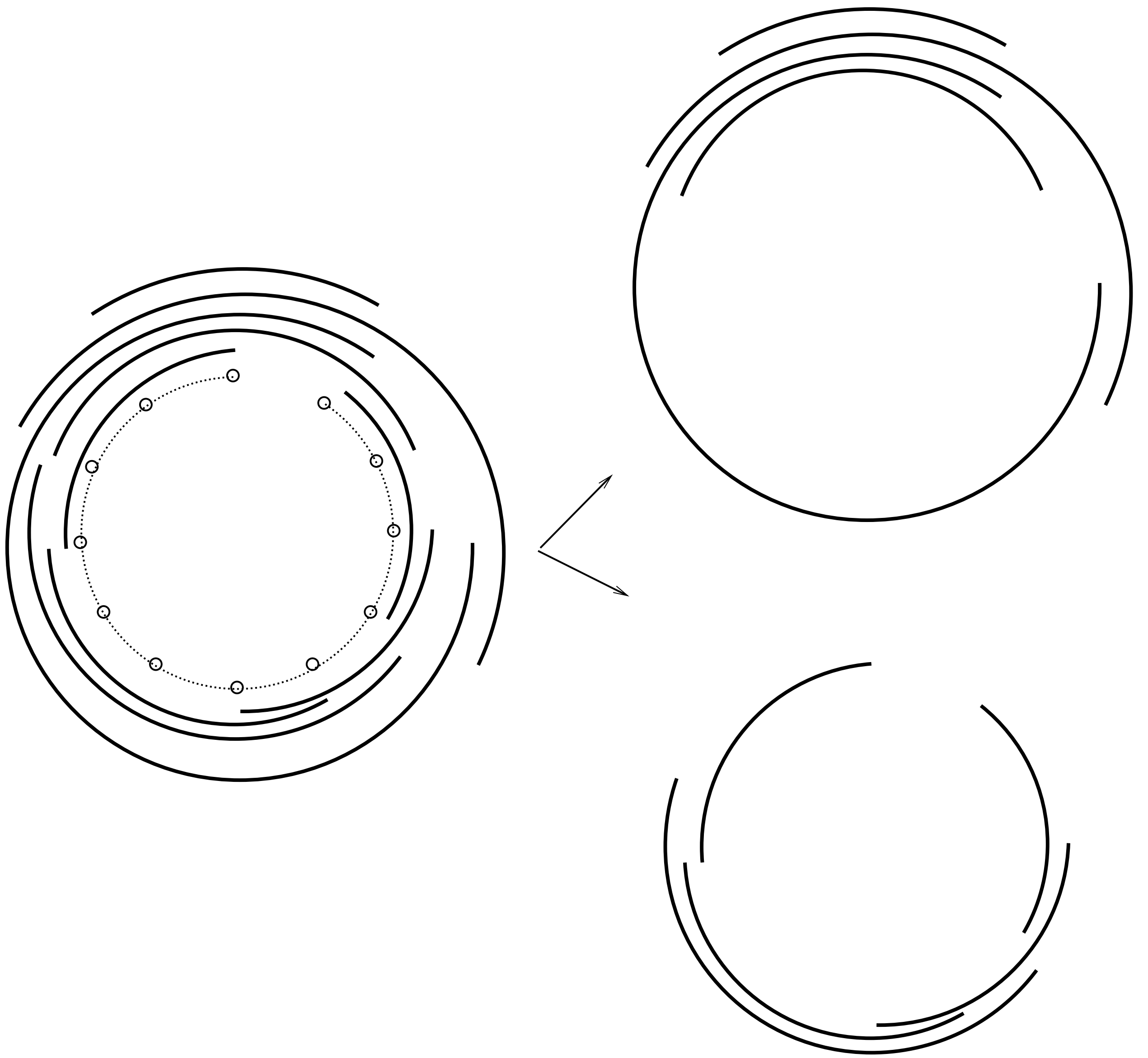
## The Chain-and-Matching Technique for Rings with Pre-Routed Requests (Simple Form)

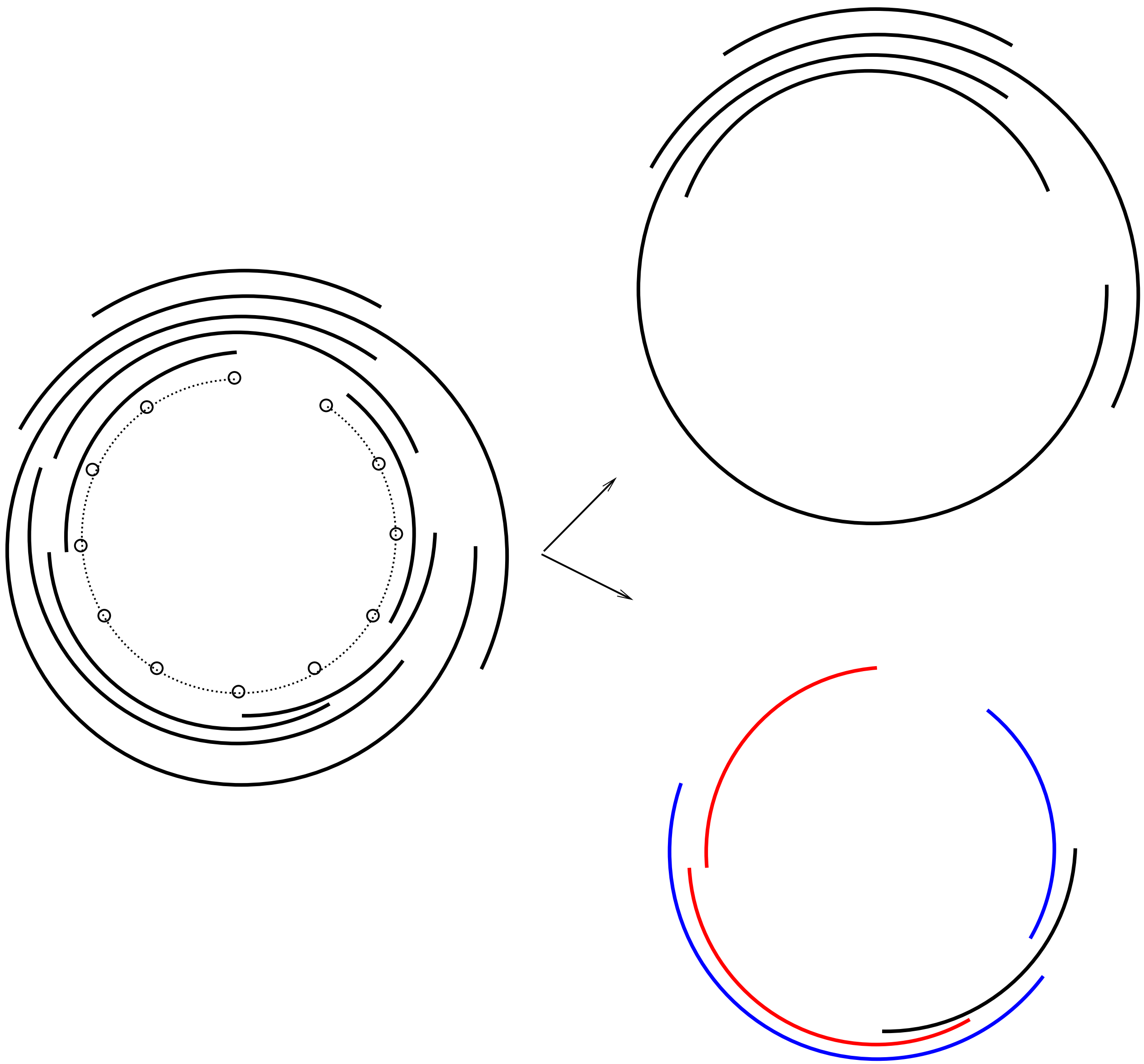
- $ALG_c$: Select a separation edge, apply chain algorithm to requests not using this edge. If colors remain unused, use them to color requests passing through the separation edge. Call this solution $SOL_c$.

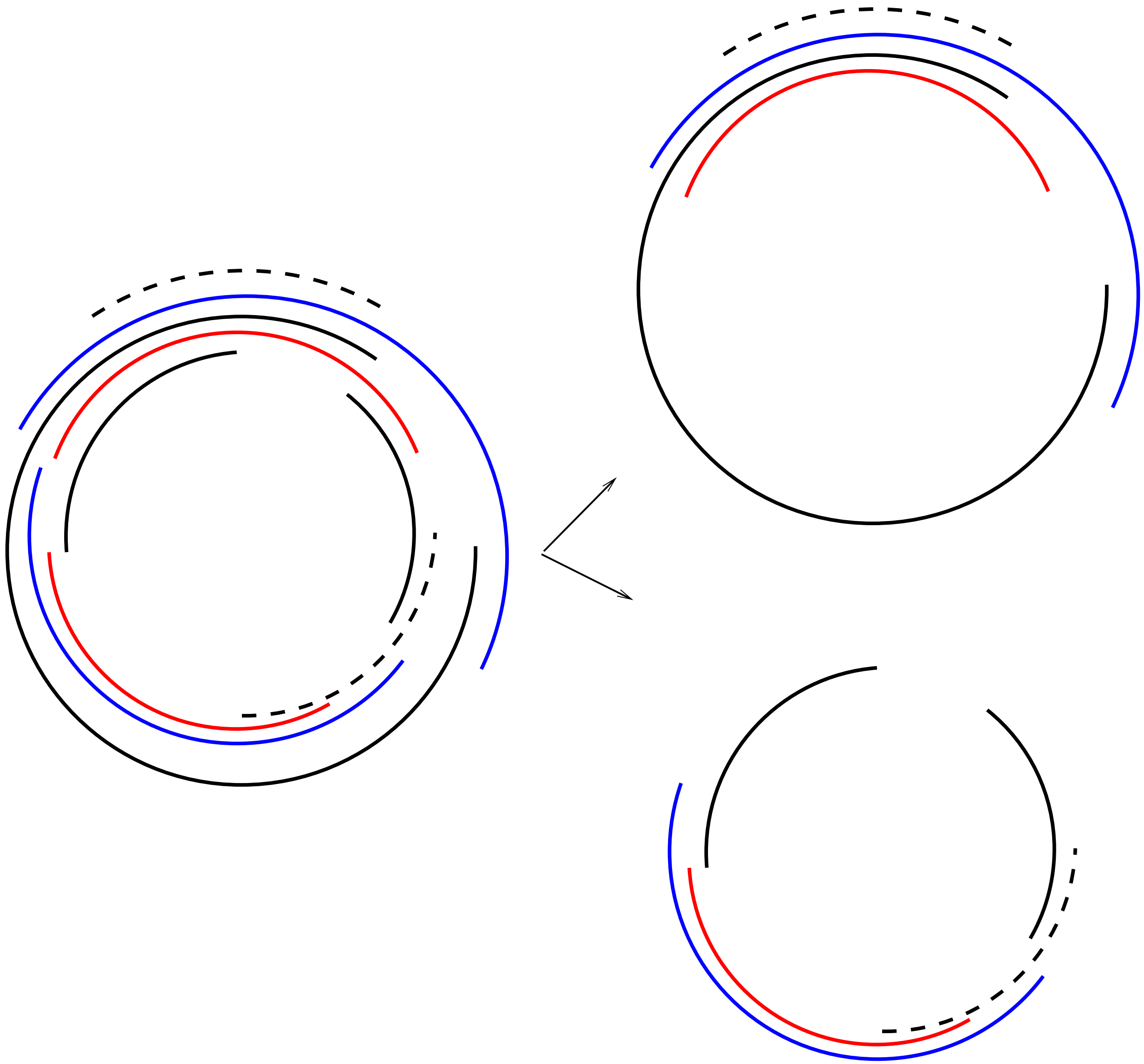
- $ALG_m$: Construct a bipartite request compatibility graph. Compute a maximum matching on this graph.

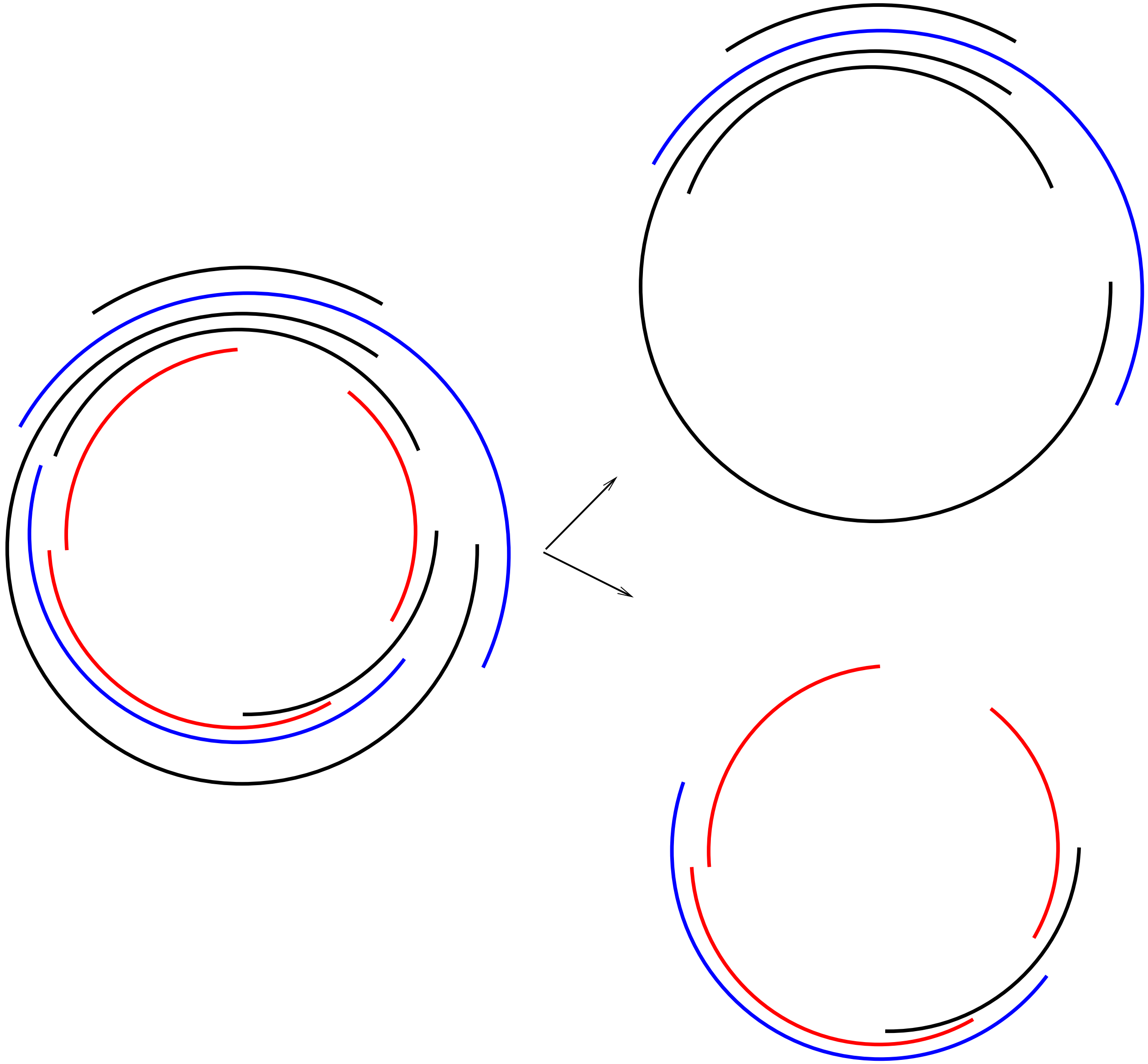
  Color requests 'from scratch' using one color for each pair of the matching, until colors or pairs finish. If pairs finish first, use the remaining colors arbitrarily. Call this solution $SOL_m$.

- $ALG$: Execute $ALG_c$ and $ALG_m$ independently, choose best solution: $SOL = \max(SOL_c, SOL_m)$.









## Analysis

$\mu$: size of the maximum matching

- $OPT \leq SOL_c + \min(w, \mu)$

  (because the chain algorithm is optimal and the only possible improvement is due to matched requests that use the separation edge – at most $w$ and at most $\mu$)

- $SOL_m \geq 2 \cdot \min(w, \mu)$

  (because pairs are colored with one color each, until there are no more pairs or no more colors)

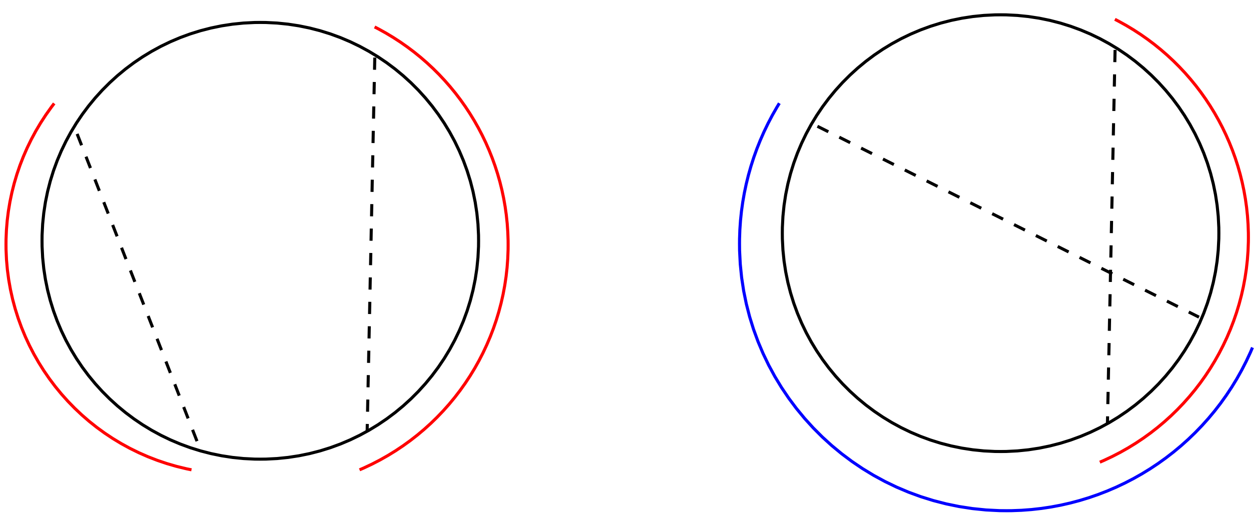
- $OPT \leq SOL_c + \frac{1}{2} SOL_m \leq \frac{3}{2} SOL$

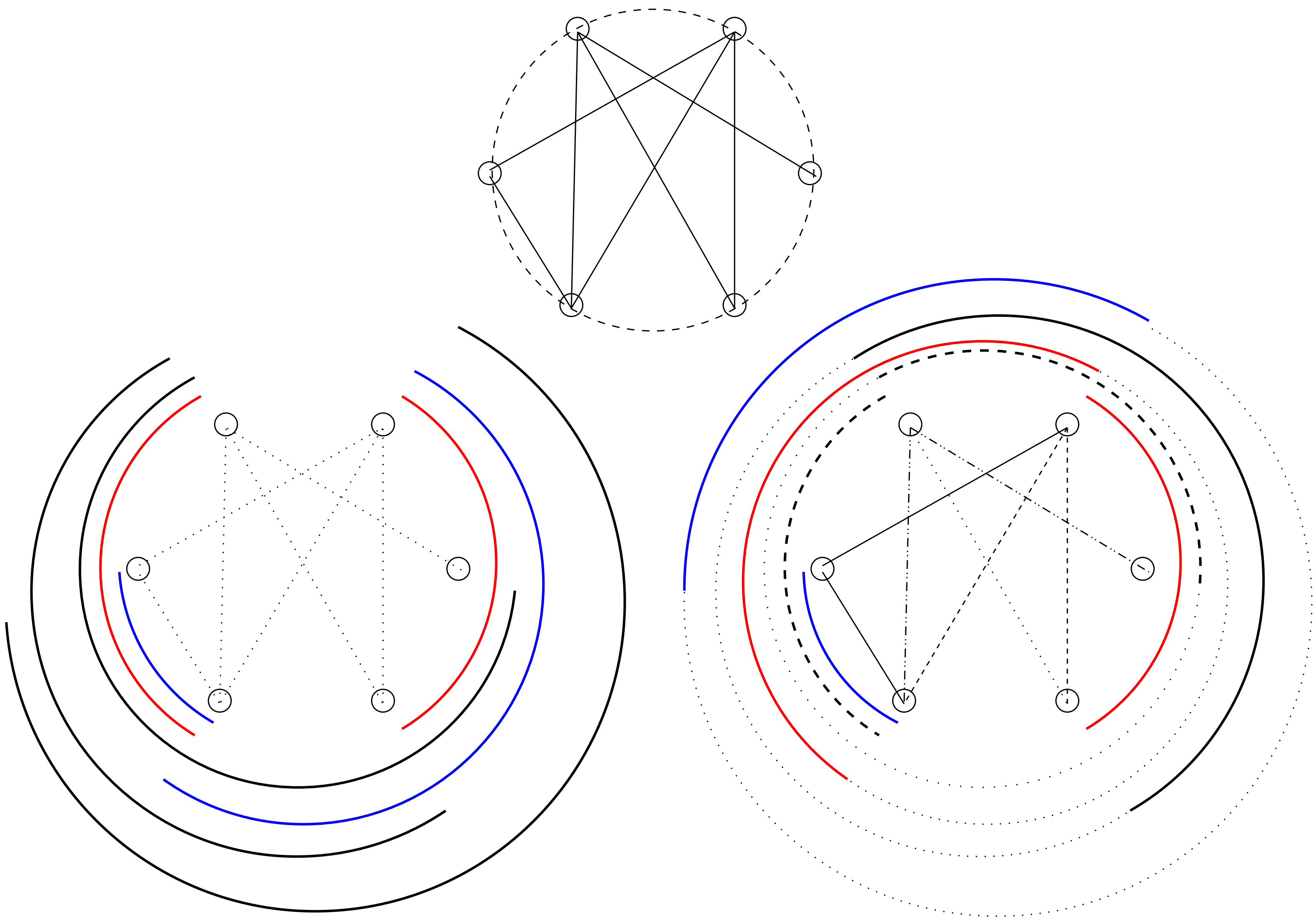
*Remark: $ALG_c$ is a 1/2-approximation algorithm for* MAXPC

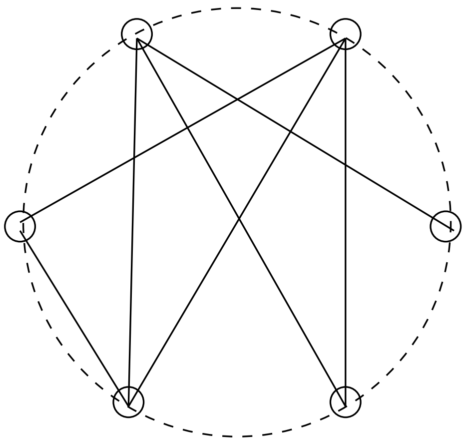
## The Chain-and-Matching Technique

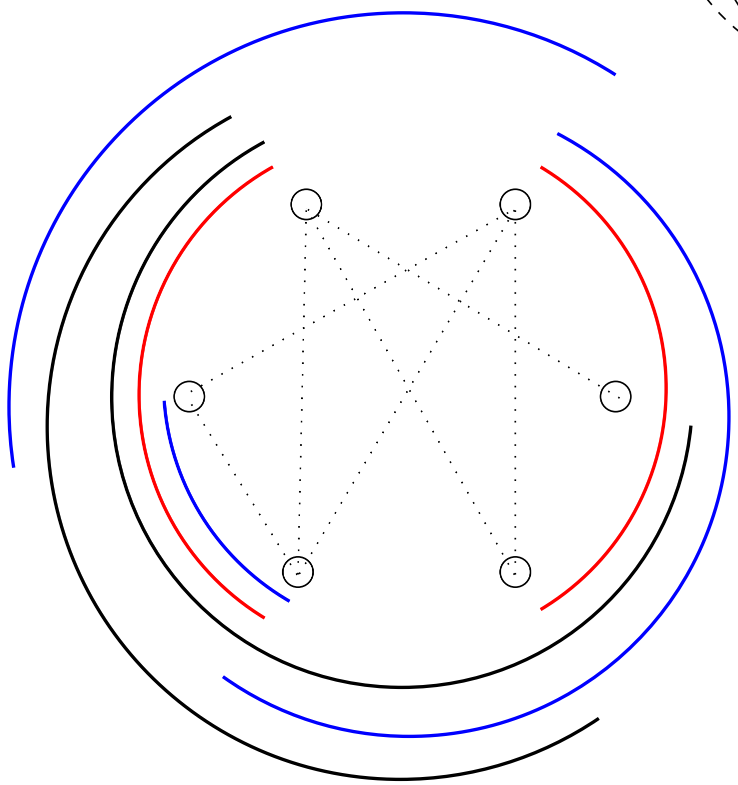
## for rings without prior routing of requests (MaxRPC)

- The chain instance consists of all requests routed so as to avoid some chosen edge.
- Compatible are now two requests that can be routed with non-intersecting paths. The compatibility graph is not bipartite any more (still it is possible to find a maximum matching efficiently).









## Analysis

$\mu$: size of the maximum matching

- $OPT \leq SOL_c + \min(w, \mu)$

  (because the chain algorithm is optimal and the only possible improvement is due to matched requests routed through the separation edge – at most $w$ and at most $\mu$)

- $SOL_m \geq 2 \cdot \min(w, \mu)$

  (because pairs are colored with one color each, until there are no more pairs or no more colors)

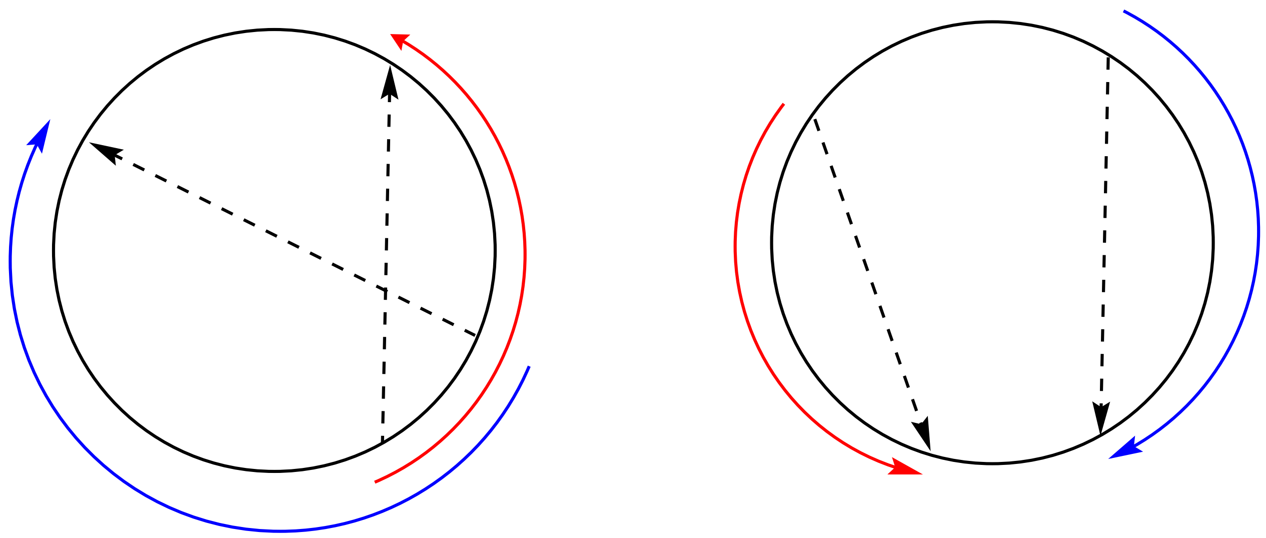
- $OPT \leq SOL_c + \frac{1}{2}SOL_m \leq \frac{3}{2}SOL$

*Remark: $ALG_c$ is a 1/2-approximation algorithm for* MAXRPC

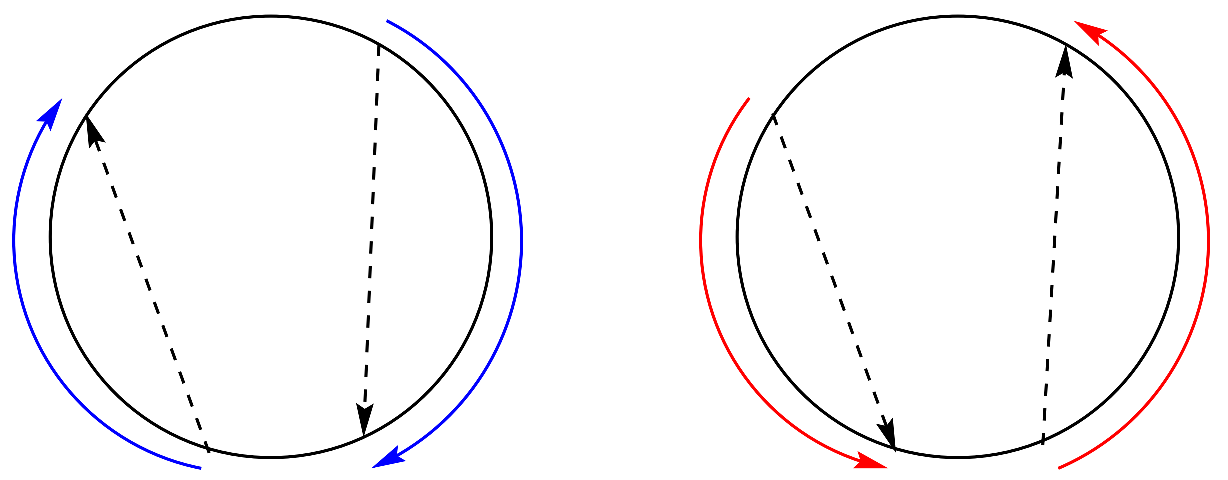
## Directed Maximum Routing-and-Path Coloring

- Requests, therefore also paths, are directed.
  Paths intersect if they use the same arc.
- Two directions: clockwise and counterclockwise.
  $w$ colors are available per direction.
- Two requests may take the same color (compatible) iff they can be routed in the same direction, using non-intersecting paths.
  Two compatible requests are either clockwise compatible or counterclockwise compatible.

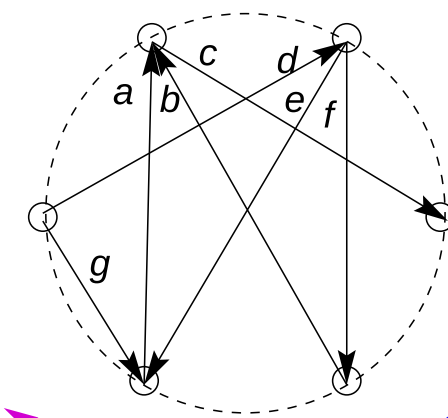
## Incompatible requests



## Compatible requests



*Clockwise* *Counterclockwise*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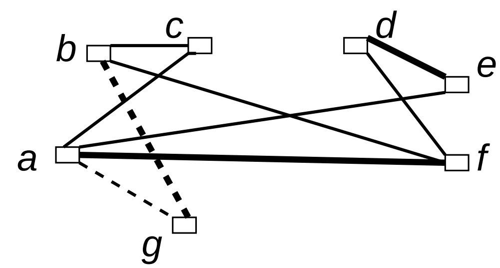
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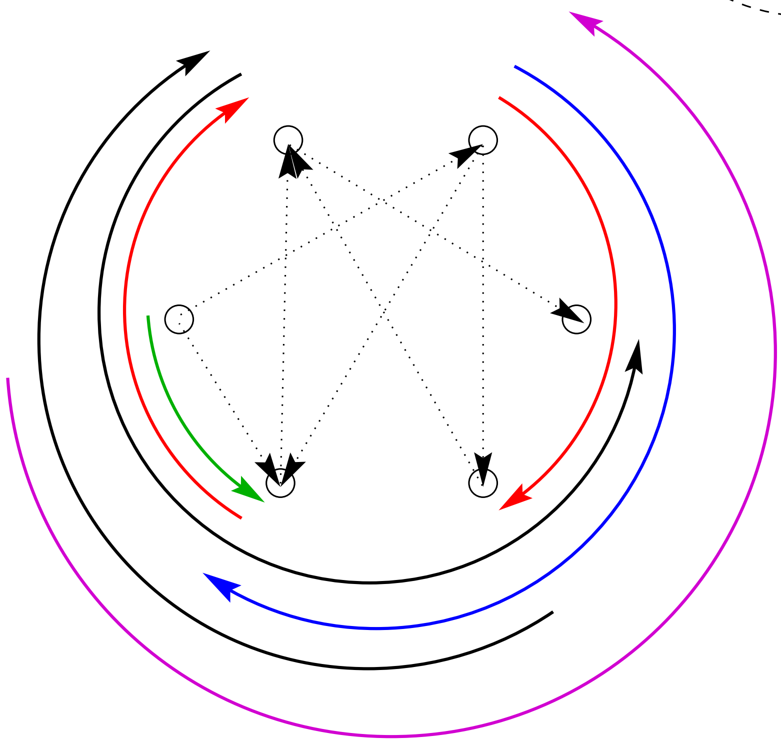

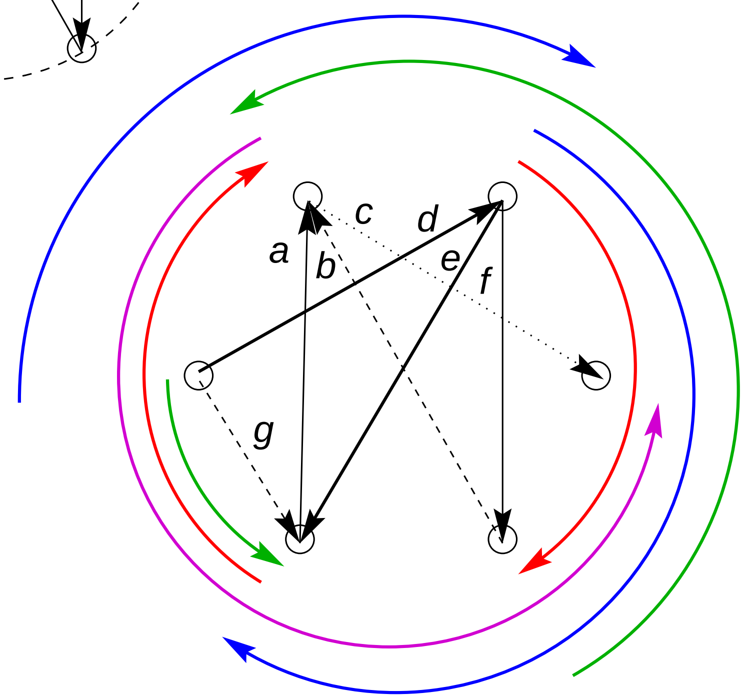
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## The $w$-BOUNDED 2-COLOR MATCHING problem

- Given is a graph $G$ with two types of edges: blue and red.
- The goal is to find a matching that maximizes

$$\min(w, \mu_r) + \min(w, \mu_b)$$

$\mu_b$ are the matched blue edges, $\mu_r$ the red ones.

*Such a matching is not necessarily maximum!*

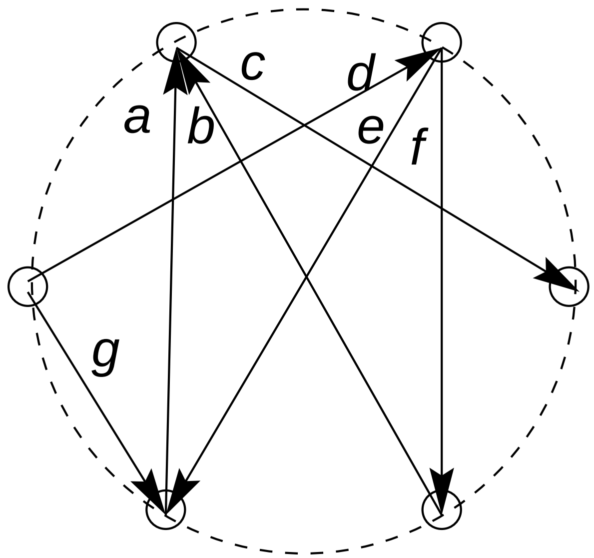
$w$-BOUNDED 2-COLOR MATCHING is NP-complete in general graphs.

Complexity open in request-compatibility graphs.

## A 3/4-approximation for $w$-Bounded 2-Color Matching

- Compute a maximum matching $M$.
  If both blue and red edges in $M$ are $\leq w$ or both are $\geq w$, stop: $M$ is a maximum $w$-bounded two-color matching.
  Otherwise:
  - Let blue edges be more than $w$, red edges fewer. Compute a maximum matching $M_r$ on the red subgraph.
  - Replace unnecessary blue edges in $M$ by red edges in $M_r$ at the cost of (at most) two blue edges for one new red.
    *Key idea: superimposing $M$ and $M_r$ we obtain a graph of degree 2, consisting of chains and cycles that alternate between $M$ and $M_r$.*

It can be shown that the cardinality of the new matching is at least $3/4$ times the cardinality of a maximum $w$-bounded 2-color matching.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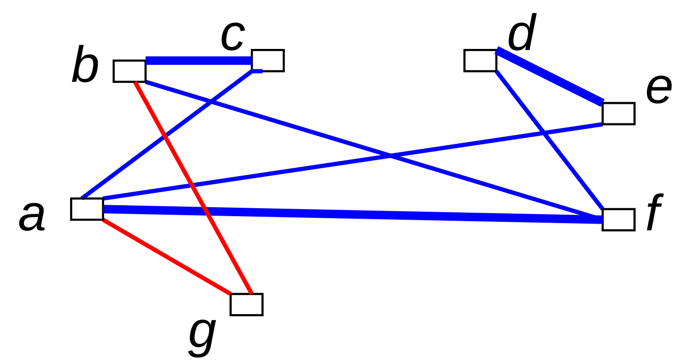
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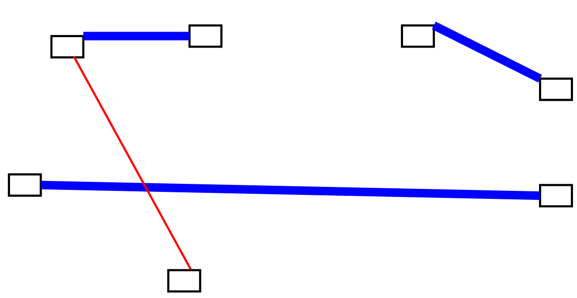

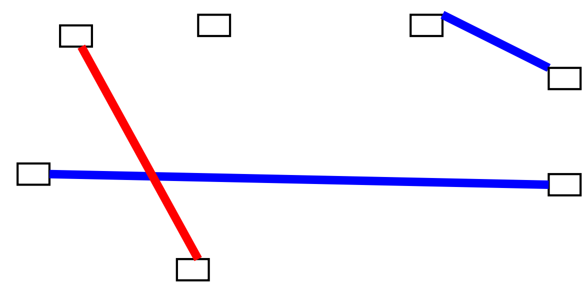

## The Algorithm for DIRECTED MAXRPC in Rings (Simple Form)

- $ALG_c$: Select a separation edge, route all requests avoiding this edge and use chain algorithm twice; Use remaining colors (if any) arbitrarily: $SOL_c$.
- $ALG_m$: Construct a request compatibility graph with two types of edges: red and blue. Compute an (as large as possible) $w$-bounded two-color matching on this graph, i.e. a matching that contains at most $w$ edges of each color.
  For each direction, color requests 'from scratch' using one color for each pair of the matching that follows this direction, until no more colors or no more pairs; if pairs finish first in some direction, use remaining colors arbitrarily: $SOL_m$.
- $ALG$: Execute $ALG_c$ and $ALG_m$ independently, choose best solution: $SOL = \max(SOL_c, SOL_m)$.

## Analysis of the algorithm for Directed MaxRPC in rings

Theorem: *Using a $\rho$-approximation algorithm for $w$-Bounded 2-Color Matching the algorithm for Directed MaxRPC in rings has approximation ratio*

$$(\rho + 1)/(\rho + 2)$$

*Proof:* We apply similar arguments as before, taking into account that there are $w$ colors *per direction*.

- Define $B_{opt}$ ($B_{sol}$): cardinality of the largest possible (algorithm's solution resp.) $w$-bounded 2-color matching.
- $OPT \leq SOL_c + B_{opt}$
- $SOL_m \geq 2 \cdot B_{sol} + 2 \cdot w - B_{sol} = 2 \cdot w + B_{sol} \geq B_{opt} + \rho \cdot B_{opt}$
- $OPT \leq SOL_c + \frac{1}{(\rho+1)} SOL_m \leq (1 + \frac{1}{\rho+1}) \cdot SOL \Rightarrow SOL \geq \frac{\rho+1}{\rho+2} \cdot OPT$

## Approximation ratio for DIRECTED MAXRPC in rings

Using the previous theorem:

- The $3/4$-approximation algorithm for W-BOUNDED 2-COLOR MATCHING, yields a $7/11 \approx 0.636$-approximation algorithm for DIRECTED MAXRPC in rings.
- Improvement: A new $7/8$-approximation algorithm for W-BOUNDED 2-COLOR MATCHING, yields a $15/23 \approx 0.652$-approximation algorithm for DIRECTED MAXRPC in rings. Also, an exact RNC algorithm leads to randomized $2/3$-apx. algorithm.

## A 0.632-approximation algorithm for DIRECTED MAXRPC in rings

- There is an algorithm that solves the problem for one color optimally: once we route a pair of requests in opposite directions in such a way that underlying paths overlap, the routing of the remaining requests is unique. We then choose a maximum subset of non-intersecting requests.
- By a known argument, using the 1-color algorithm repeatedly $w$ times we obtain a $(1-(1-1/w)^w)$-approximation algorithm for the problem with $w$ colors. For large $w$ this gets close to $(e-1)/e \approx 0.632$.
- Comparison: the $15/23 \approx 0.652$-algorithm gives a better approximation guarantee for large $w$:

$$\forall w \geq 10, \quad 1-(1-\frac{1}{w})^w \leq \frac{15}{23}$$

## Conclusions - Open Problems

- We have obtained 2/3-approximation for undirected MaxRPC, 15/23 for the directed case, and 2/3 whp. (*Note also: all chain algorithms achieve 1/2-approximation.*)
- Further improvements [Caragiannis'07]: 3/4 for undirected MaxRPC, $\approx 0.7$ for directed case. Results extend to MaxPC; $\approx 0.67$ for weighted MaxPC.
- Can these ratios be further improved? Upper bounds? PTAS?
- Other topologies: trees, trees of rings, meshes?
- Model variations: weighted requests, (full/limited) wavelength conversion, on-line, multiple fibers (full/limited).