On the Unique Games Conjecture

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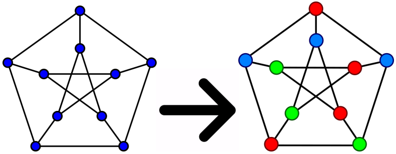
Overview

Graph 3-colorability

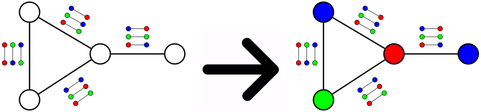
- We are given a graph $G(V, E)$.
- Can we paint the nodes of G using 3 colors in a way that all adjacent nodes have a different color?
- More formally, is there a function $f : V \to \{1, 2, 3\}$ such that, $\forall (u, v) \epsilon E : f(u) \neq f(v)$?



- The decision problem is *NP*-Complete for 3 colors or more. However, with 2 colors it's easy to find a solution as each node's color specifies its neighbours' color.
- What if we add some constraints so we can achieve the same result for more than 2 colors?

A seemingly unimportant problem

- Now our graph $G(V, E)$ has certain constraints for each node over an alphabet of size k (the colors).
- As before we are trying to find an assignment of colors that satisfies the constraints (that are now different for each node).



- The decision problem is obviously in P.
- What happens if we only consider no-instances and try to find the maximum value of an instance?
- The holy grail of approximation problems!

Let's consider first the notion of approximation algorithms and inapproximability as well as the PCP theorem.

- Let I denote an NP-complete problem with input size N and let $OPT(I)$ denote the value of the optimal solution
- We define $ALG(I)$ to be the value of the solution that a polynomial time approximation algorithm finds for I.

C-approximation algorithm

We say that the algorithm achieves an approximation factor of C if on every instance I,

- $ALG(I) \geq OPT(I)/C$ *if I is a maximization problem*,
- $ALG(I) \leq C \cdot OPT(I)$ *if I is a minimization problem*.

Inapproximability

- A maximization problem is proved to be inapproximable by giving a reduction from an *NP*-complete problem such as $\mathrm{3SAT}$ to a gap version of I.
- A (c, s)-gap version of I, denoted $GapI_{c,s}$, is a promise problem where either $OPT(I) \geq c$ or $OPT(I) \leq s$.
- Suppose there is a polynomial reduction from $\mathrm{3SAT}$ to $GapI_{c,s}$ for some $0 < s < c$, that maps a $\mathrm{3SAT}$ formula ϕ to an instance $\mathcal{I}$ of I such that:
    - If ϕ has a satisfying assignment, then $OPT(\mathcal{I}) \geq c$.
    - If ϕ has no satisfying assignment, then $OPT(\mathcal{I}) \leq s$.
- If there were an algorithm with approx. factor strictly less than c/s for problem I, then it would enable us to efficiently decide whether a $\mathrm{3SAT}$ formula is satisfiable.

The PCP Theorem 1/2

A reduction like the above in fact a sequence of reductions. The first one is the famous *PCP Theorem* which can be defined as a reduction from $3\mathrm{SAT}$ to a gap version of $3\mathrm{SAT}$.

- For a $3\mathrm{SAT}$ formula ϕ, let $OPT(\phi)$ denote the maximum fraction of clauses that can be satisfied by any assignment.
- Thus $OPT(\phi) = 1$ iff ϕ satisfiable.

PCP Theorem

There is a universal constant $\alpha < 1$ and a polynomial time reduction that maps a $3\mathrm{SAT}$ instance ϕ to another $3\mathrm{SAT}$ instance ψ such that:

- If $OPT(\phi) = 1$, then $OPT(\psi) = 1$.
- If $OPT(\phi) < 1$, then $OPT(\psi) \leq 1$.

The PCP Theorem 2/2

Alternatively, the *PCP Theorem* can be formulated in terms of proof checking.
In particular, the theorem states that every *NP* statement has a polynomial size proof that can be checked by a probabilistic polynomial time verifier by reading only a constant bits in the proof.
The verifier has the completeness and the soundness property, meaning that every correct statement is accepted with probability 1 and every incorrect statement is accepted with a very small probability, e.g. 1%.

Label Cover / Unique Game

- Let's consider now the Label Cover problem again and formulate a generalization of it.
- That is the *Unique Game*.

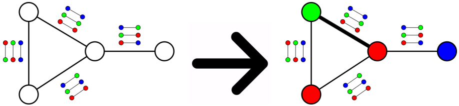
Unique Game Definition

A Unique Game $\mathcal{U}(G(V,E),[n],\{\pi_e | e \in E\})$ is a constraint satisfaction problem defined as follows: $G(V,E)$ is a directed graph whose vertices represent variables and edges represent constraints. The goal is to assign to each vertex a label from the set $[n]$. The constraint on an edge $e=(v,w) \in E$ is described by a bijection $\pi_e : [n] \mapsto [n]$. A labeling $L : V \mapsto [n]$ satisfies the constraint on edge $e=(v,w)$ iff $\pi_e(L(v)) = L(w)$. Let $OPT(\mathcal{U})$ denote the maximum fraction of constraints that can be satisfied by any labeling:

$$OPT(\mathcal{U}) := \max_{L:V\mapsto[n]} \frac{1}{|E|} \cdot |\{e \in E \mid L \, satisfies \, e\}|.$$

an Example

Let's first see an example:



The value of this instance is $3/4$ as it satisfies all the edges except the thick one.
We can now make some observations:

- We only consider instances with value less than 1 as it's very easy to find a solution otherwise.
- Can we easily differentiate instances that have a value of $3/4$ like the one above from instances that have a value of $1/4$?

Unique Games Conjecture

Unique Games Conjecture Definition

For every $\epsilon, \delta > 0$, there exists a constant $n = n(\epsilon, \delta)$, such that given a Unique Game instance $\mathcal{U}(G(V, E), [n], \{\pi_e | e \in E\})$, it is NP-hard to distinguish between these two cases:

- YES Case: $OPT(\mathcal{U}) \geq 1 - \epsilon$.
- NO Case: $OPT(\mathcal{U}) \leq \delta$.

- For every ϵ, δ there exists a sufficiently large for which the conjecture holds on Unique Games with label size n.

Remarks and a variant of the Conjecture

Some more remarks:

- For the conjecture to hold there must be $n \geq \frac{1}{\delta}$ as every assignment satisfies $\frac{1}{n}$ of the constraints. Also, it is known that $n \geq 2^{\Omega(\frac{1}{\epsilon})}$.
- The gap problem must be computationally hard (no polynomial time algorithm) and not necessarily NP-hard.

Weak Unique Games Conjecture

There is an increasing unbounded function $\Gamma : \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that the following holds: for every $\epsilon > 0$, there exists a constant $n = n(\epsilon)$ such that given a Unique Game instance $\mathcal{U}(G(V, E), [n], \{\pi_e | e \in E\})$, it is NP-hard to distinguish between these two cases:

- YES Case: $OPT(\mathcal{U}) \geq 1 - \epsilon$.
- NO Case: $OPT(\mathcal{U}) \leq 1 - \sqrt{\epsilon} \cdot \Gamma(1/\epsilon)$.

Inapproximability Results 1/2

- Why do we care about UGC?
- UGC can be used to make reductions of *NP*-complete problems to *GapUG* and provide optimal results for open problems in the field of approximation algorithms.
- In fact, it affects a lot of famous problems.

Let's consider the following problems:

- Vertex Cover: While the best approximation known is $2 - approx.$ there is the possibility for $1.36 - approx.$ Under the UGC that limit goes up to $2 - \epsilon$.
- Max Cut: Best approx by Goemans and Williamson achieves $\alpha \approx 0.878$ while it is not proven to be *NP*-hard to approximate all the way to $\frac{16}{17} \approx 0.941$. UGC makes 0.878 optimal.
- Sparsest cut, Min-2SAT-Deletion and more.

Inapproximability Results 2/2

Problem	Best Approx. Known	Inapprox. Known Under UGC	Best Inapprox. Known
Vertex Cover (VC)	2	$2-\varepsilon$	1.36
VC on k-uniform Hypergraphs, $k \geq 3$	k	$k-\varepsilon$	$k-1-\varepsilon$
MaxCut	α_{MC}	$\alpha_{\text{MC}}-\varepsilon$	$\frac{17}{16}-\varepsilon$
Max-2SAT*	α_{LLZ}	$\alpha_{LLZ}-\varepsilon$	APX-hard*
Any CSP $\mathcal{C}$ with integrality gap $\alpha_{\mathcal{C}}$	$\alpha_{\mathcal{C}}$	$\alpha_{\mathcal{C}}-\varepsilon$	
Max-kCSP	$O(2^k/k)$	$\Omega(2^k/k)$	$2^{k-O(\sqrt{k})}$
Max-3CSP on satisfiable instances	$\frac{8}{5}$	$\frac{8}{5}-\varepsilon$, under Conj. 3.6	$\frac{27}{20}-\varepsilon$
Max Acyclic Subgraph	2	$2-\varepsilon$	$\frac{66}{65}-\varepsilon$
Feedback Arc Set	$\tilde{O}(\log N)$	$\omega(1)$	APX-hard
Non-uni. Sparsest Cut	$\tilde{O}(\sqrt{\log N})$	$\omega(1)$	APX-hard
Uniform Sparsest Cut,	$O(\sqrt{\log N})$	$\omega(1)$, under Hypo. 3.4	No PTAS
Min-2SAT-Deletion, Min-Uncut	$O(\sqrt{\log N})$	$\omega(1)$	APX-hard
Coloring 3-colorable Graphs	$N^{.2111}$	$\omega(1)$, under Conj. 3.7	5
Coloring $2d$-colorable Graphs, $d \geq 2$	$N^{1-\frac{3}{2d+1}}$	$\omega(1)$, under Conj. 3.6	$2d+2\lfloor\frac{2d}{3}\rfloor-1$, $d^{\Omega(\log d)}$

Algorithms for the Unique Game 1/4

- The UGC states that given an instance of the Unique Game problem that is $1 - \epsilon$ satisfiable, it is *NP*-hard to find an assignment satisfying δ fraction of the constraints when the number of labels $n(\epsilon, \delta)$ is a sufficiently large constant.
- As a result, all it takes to disprove the UGC is a good enough algorithm.

Let's see a summary of the algorithmic results so far:

Algorithm	**Value of Solution Found on $1-\varepsilon$ satisfiable instance with n labels and N vertices**
Khot [60]	$1 - O(n^2 \varepsilon^{1/5} \sqrt{\log(1/\varepsilon)})$
Trevisan [103]	$1 - O(\sqrt[3]{\varepsilon} \log N)$
Gupta, Talwar [46]	$1 - O(\varepsilon \log N)$
Charikar, Makarychev, Makarychev [22]	$1 - O(\sqrt{\varepsilon} \log n)$ $n^{-\varepsilon/2}$
Chlamtac, Makarychev, Makarychev [30]	$1 - O(\varepsilon \sqrt{\log N \log n})$
Arora *et al* [7]	$1 - O(\varepsilon \cdot \frac{1}{\lambda} \log(\frac{\lambda}{\varepsilon}))$ on graphs with eigenvalue gap λ.
Arora, Barak, Steurer [5]	$1 - \varepsilon^{\alpha}$ for some $0 < \alpha < 1$ in time $\exp(N^{\varepsilon^{\alpha}})$.

Algorithms for the Unique Game 3/4

- None of the above algorithms manages to disprove the conjecture for various reasons.
- For example, [60] and [22] do not work when n is sufficiently large.
- [103], [46], [30] work only when ϵ is sub-constant function of the instance size N.
- [7] works only in mild expander graphs and [5], one of the most recent results, runs in sub-exponential time.

Algorithms for the Unique Game 4/4

Almost all of the algorithms depend on a natural SDP (Semi Definite Programming) relaxations for the Unique Game Problem. In particular:

$$\text{Maximize} \quad \frac{1}{|E|} \sum_{e=(v,w)\in E} \sum_{i\in[n]} \left\langle \mathbf{x}_{v,i}, \mathbf{x}_{w,\pi_e(i)} \right\rangle$$

Subject to,

$$\forall v \in V \qquad \sum_{i\in[n]} \|\mathbf{x}_{v,i}\|^2 = 1.$$

$$\forall v \in V,\ i,j \in [n],\ i \neq j \qquad \langle \mathbf{x}_{v,i}, \mathbf{x}_{v,j} \rangle = 0.$$

$$\forall v, w \in V,\ i,j \in [n] \qquad \langle \mathbf{x}_{v,i}, \mathbf{x}_{w,j} \rangle \geq 0.$$

Will the conjecture hold?

The academic world seems about evenly divided on whether it is true or not.

Arguments in favor include:

- Weak Unique Games Conjecture is a priori weaker than UGC and seems more believable.
- The SDP Relaxation shown for the Unique Game Problem has an $(1-\epsilon, \delta)$ integrality gap.

Arguments Against:

- There is no knowledge for any other natural problem equivalent to Unique Game problem.
- A certain type of SDP, Lasserre SDP, may disprove the UGC.

A win-win situation

If the conjecture is proven:

- Great number of inapproximability results finalized.
- Geometry and analysis connections unified.

On the other hand, if it's disproven:

- Almost certain algorithmic breakthrough, a proof that the conjecture doesn't hold would need an algorithm with many new ideas probably beyond the barrier of SDP programming.
- Open field for better approximation algorithms for many important problems.

Either way, it's certain that the conjecture will lead to more new techniques and results and enrich the theory in the fields it has applications.