Online Algorithms

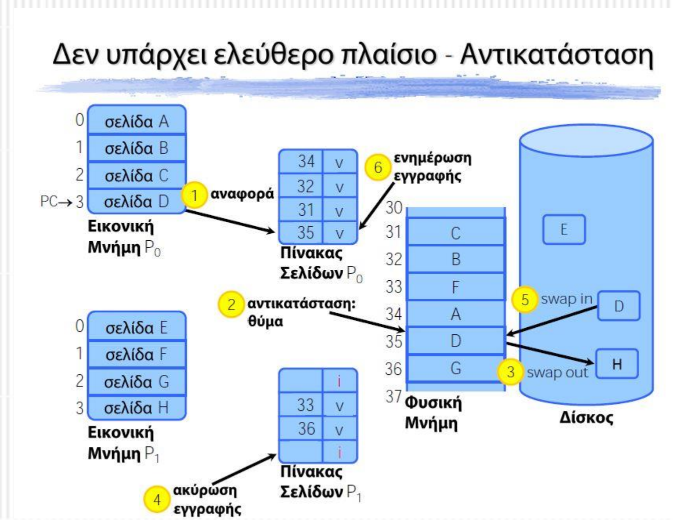
- Theory of Computation
- School of Electrical and Computer Engineering
- National Technical University of Athens

Επιμέλεια Διαφανειών: Σπύρος Τσουκαλάς

Types of Problems

- For certain problems, input is not available from the beginning

- Certain decisions are requested on the way
  - Output required



Online vs Offline

- Online Algorithms
  - Input arrives as sequence of input portions
  - The system must react in response to request
  - Future input is unknown
  - Not optimal

- Offline Algorithms
  - Entire input is given in advance
  - Solve the problem at hand
  - Future is known
  - Optimal

Competitive Analysis

- Big O complexity can’t be used:
  - For every algorithm there will be a sequence that makes it look foolish
- Competitive Ratio
  - Comparison with an optimal offline algorithm processing the same sequence of requests
  - Maximum cost over all possible input sequences divided by the cost of an optimal offline algorithm
  - Related to minimax concept of game theory
    - Online player vs Adversary

Competitive Analysis

- A little formalism:
  - $cost_A(\sigma)$: the cost of an online algorithm A on the input sequence σ
  - $cost_{OPT}(\sigma)$: the cost of the optimal offline algorithm on σ

- Algorithm A is c competitive if there exists a constant b such that on every request sequence σ:

$$cost_A(\sigma) \ \le \ c \cdot cost_{OPT}(\sigma) \ + \ b$$

The Ski Rental Problem

- Cost for renting a pair of skis
- Cost for buying a pair of skis

- Rent or Buy? When?
  - How do we decide?

- Request = “Take a ski trip”
- Actions = “rent” | “buy” | “use skis already bought”
- Costs = 1, s, 0 respectively

- On a sequence of t requests any sensible online algorithm is of the form:

  “Rent for the first k trips, then buy, then use already bought”

The Ski Rental Problem

- Online Cost
  - t, $t \leq k$
  - k+s, $t > k$
- Offline Cost
  - min(s, t)

- Find k that minimizes the competitive ratio.
- For given k, k+1 maximizes the ratio

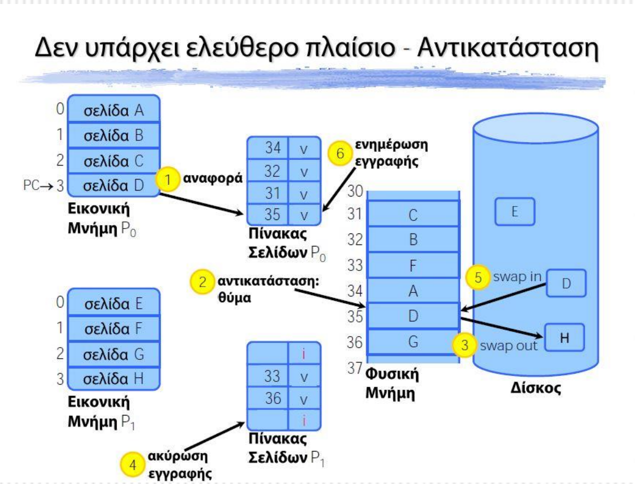
$$\frac{k+s}{\min(k+1,\ s)}$$

- For given k, k+1 requests maximize the ratio. The ratio is minimized for $k = s - 1$

- The on-line player should rent until enough ski trips have occurred so that he would have done better if he had bought skis initially

Paging

- Memory management scheme
- Memory hierarchy
- Page fault minimization



- Set of n pages
- RAM with capacity for k pages
- The system receives requests for pages in RAM
- If the referenced page is in the RAM, the request can be served
- If not, then a page fault occurs
- The missing page is loaded from secondary storage and an online algorithm has to decide which page to evict

Paging

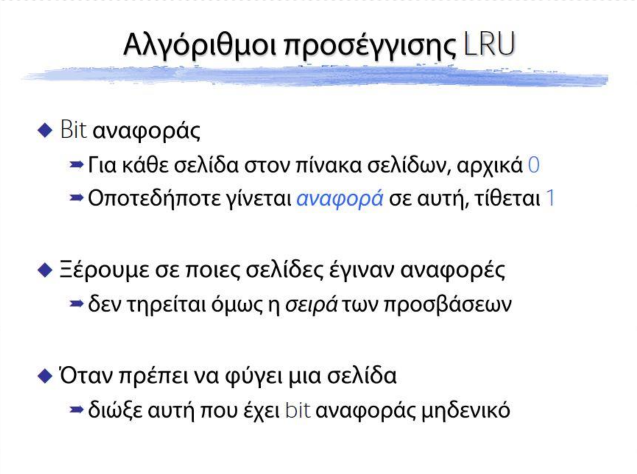
- Common algorithms
  - LRU: evict the page in memory that was requested least recently
  - FIFO: evict the page that has been longest in memory

- Theorem
  - FIFO and LRU are k-competitive, where k the size of main memory in pages

- There exists a more general class of algorithms that achieve a competiveness of k

Paging

- Marking
  - Each page is associated with a bit called mark
  - Initially all pages are set as unmarked
  - Stages of page requests
  - A page is marked when it is first requested in this stage
  - On a fault, an unmarked page is evicted



- Theorem
  - Any marking algorithm is k-competitive

Paging

- Theorem
  - No deterministic online algorithm for the paging problem can achieve a ration smaller than k

- Proof
  - Optimal Offline Algorithm
    - Belady's greedy algorithm
    - "Sees" in the future
    - On a fault, evict the page whose next request occurs furthest in the future

Paging

- Proof
  - A and OPT start with the same set of pages in memory
  - The adversary restricts its request sequence to a set of k+1 pages, the pages initially in the memory and another one
  - It always requests the page that is outside of the memory
  - This can be continued for an arbitrary number of requests, resulting in a sequence σ on which A faults on every request

  - What remains is to show that $cost_{OPT}(\sigma) \leq \left\lceil \frac{|\sigma|}{k} \right\rceil$

  - At each fault, the adversary evicts the page whose first request occurs furthest in the future
  - The adversary is guaranteed that there will be at least k-1 pages requested between any two faults, so the adversary faults at most on every k^{th} request

Adversaries

- Online algorithms can achieve better performance if they are allowed to make random choices
- The competitive ratio of a randomized algorithm is defined with respect to an adversary

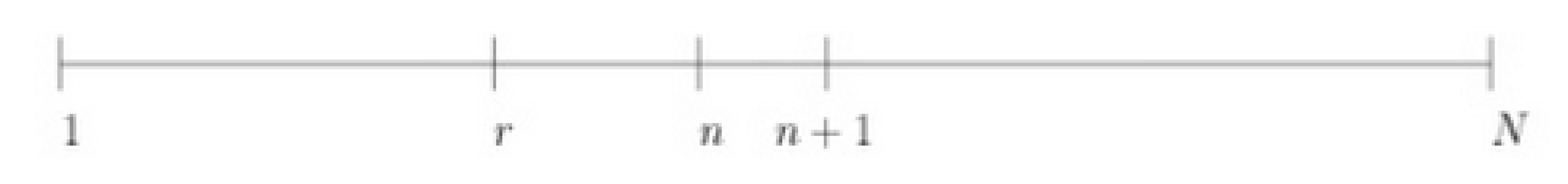
- There are three types of adversaries:
  - oblivious adversary (weak)
    - generates the whole request in advance
  - adaptive online adversary (medium)
    - it may observe the online algorithm and generate next request based to all previous requests
  - adaptive offline adversary (strong)
    - knows everything. Even randomization can't face it

Secretary Problem

- Also known as the marriage problem, the game of googol
- There is a single secretarial position to fill
- There are n applicants for the position
- The applicants can be ranked from best to worst unambiguously
- The goal is to have the highest probability of selecting the best applicant of the whole group
- They are interviewed sequentially in random order
- Immediately after the interview, the applicant is either accepted or rejected irrevocably

Secretary Problem

- Strategy
  - Naive: pick the ith candidate: $P(Success) = \frac{1}{N}$
  - Interview the first r applicants for r<n
  - Accept the very next applicant that is better than all the first r you interviewed



  - A=n+1 the best applicant, r the last that will be rejected

Secretary Problem

- Strategy
  - A won't be chosen, unless:
    - $n \geq r$
    - The highest applicant in [1,n] is the same as in [1,r], $P = \frac{r}{n}\frac{1}{N}$

$$P(Success) = P(r) = \frac{1}{N}\left[\frac{r}{r} + \frac{r}{r+1} + \cdots + \frac{r}{N-1}\right] = \frac{r}{N}\sum_{n=r}^{N-1}\frac{1}{n}$$

  - For the optimal solution, P'(r)=0 => $\mathrm{r} = \frac{1}{e} \approx 0.37$
  - Coincidentally, $\mathrm{P}(r_{max}) = \frac{1}{e}$

Applications and Further Research

- Stock Markets
  - Algorithms for stock prediction

- Large networks
  - Network switches
  - TCP Acknowledgement

- Robot Motion Planning

- Bin Packing

- Storage Allocation and Cache Management

- Job Scheduling

Bibliography

- On-Line Algorithms Versus Off-Line Algorithms: How Much it Worth to Know the Future? – Richard Karp, 1992
- Online Algorithms – Susanne Albers
- Competitive Analysis of Paging: A Survey – Sandy Irani
- Algorithm Design – Kleinberg, Tardos

Questions?

Ευχαριστώ