Generalized Bipartite Matching

Peli Teloni

Network Algorithms
$\mu\Pi\lambda\forall$

July 22, 2014

Outline

Real-Life situation

Bob

Real-Life situation

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e-mail from DVD-rental store

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wants to watch
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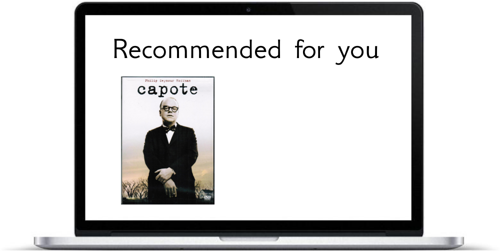

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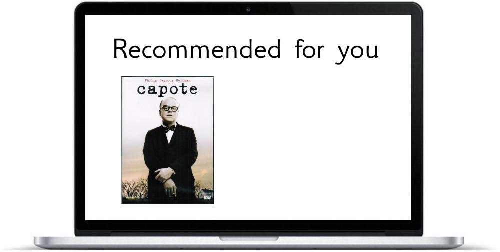

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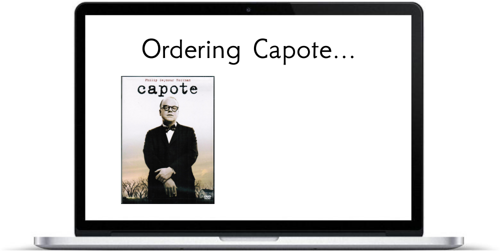

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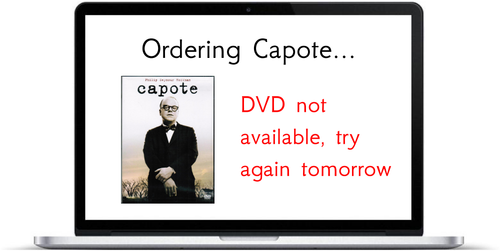
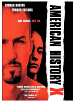
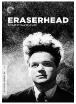


A Simple Recommender System

			
	?	?	2
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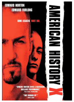

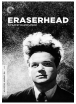

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Step 1: Predict preferences

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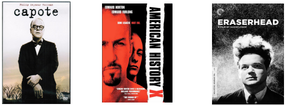

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Step 1: Predict preferences

Step 2: Recommend preferred movies

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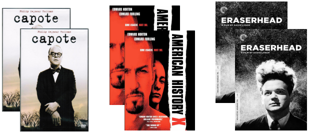

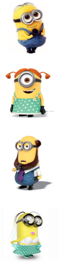

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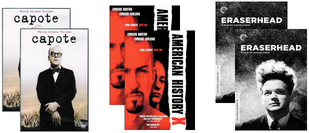

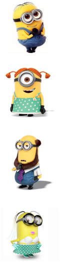

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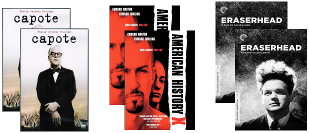

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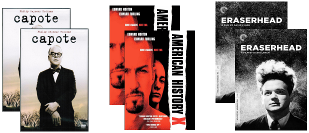

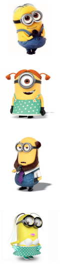

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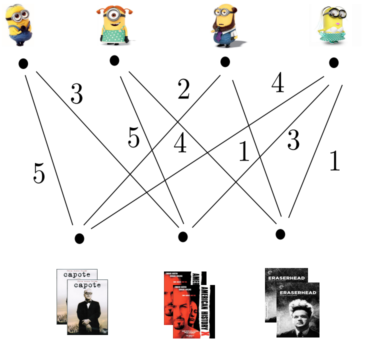

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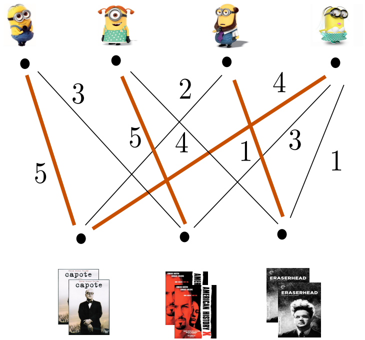

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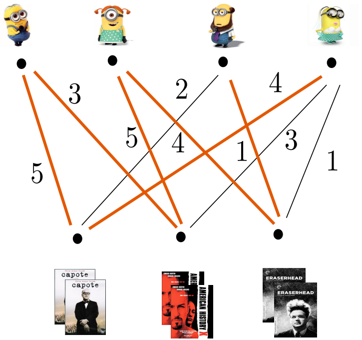

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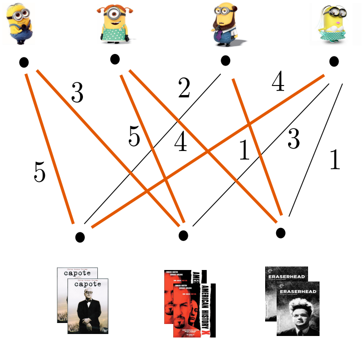

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Generalized Bipartite Matching

Input: a bipartite weighted graph $G = (U, V, E)$ with degree constraints

Output: a max-weight subset of edges that satisfy these constraints

Solving GBM

- Optimal solution in poly-time
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  - at present: Netflix has 20M users, 10k's of movies
- In this paper: near-optimal solution
  - poly-log time
  - strong approximation guarantees
  - first distributed algorithm

Overview for GBM_ϵ

Definition of GBM_ϵ

Given $\epsilon > 0$ and an instance of GBM, if GBM is feasible then find $\bar{E} \subseteq E$ s.t.:

$$\lfloor (1-\epsilon) l(v) \rfloor \leq |\bar{E}_v| \leq \lceil (1+\epsilon) b(v) \rceil \quad \forall v \in U \cup V$$

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Step 2: Rounding

- outputs actual matching
- distributed dependent
- keeps approx. guarantees

Outline

MPC-LP

$$\begin{aligned} \max\ & w^T x \\ \text{s.t.}\ & Px \le p \\ & Cx \ge c \\ & x \ge 0 \end{aligned}$$

- subclass of LPs
- non negative coefficients/variables
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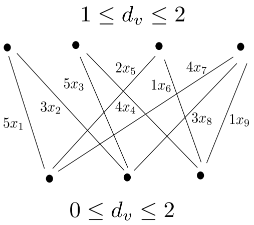
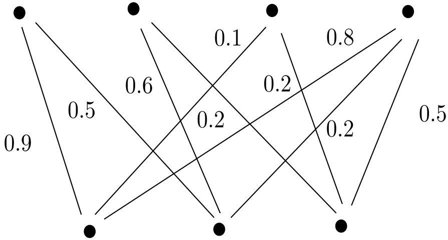


- subclass of LPs
- non negative coefficients/variables
- facility location, circuit routing etc.
- LP_{GBM} is an example of LP_{MPC}
  - constraints: Ms
  - variables and weights: Bs

$$\begin{aligned} \text{max } & \sum_{e \in E} w(e) x_e \\ \text{s.t. } & \sum_{e \in E_v} x_e \le b(v) \\ & \sum_{e \in E_v} x_e \ge l(v) \\ & x_e \in \{0, 1\} \end{aligned}$$

MPCSolver

$$1 \le d_v \le 2$$



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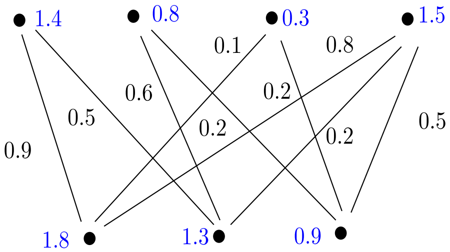

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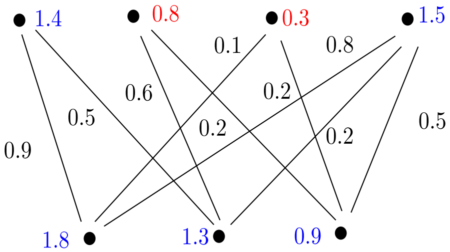

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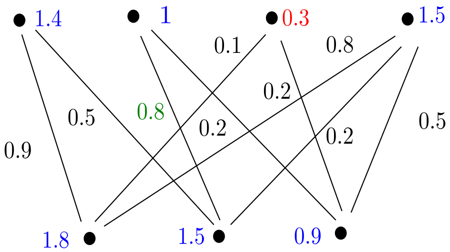

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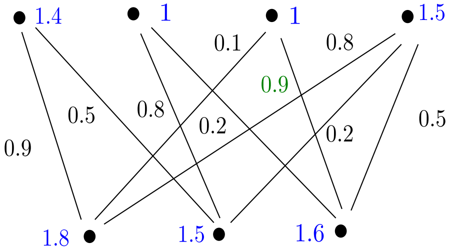

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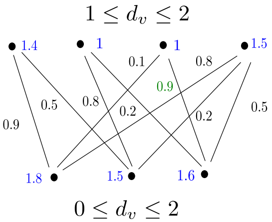


repeat:

compute $y_i(x) = \exp(\mu(P_i x - 1))$ for $i \in [m]$

compute $z_i(x) = \exp(\mu(1 - C_i x))$ for $i \in [k]$

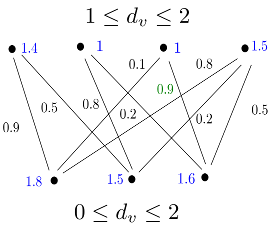
for $j = 1, \ldots, n$

if $\frac{P_j^T y}{C_j^T z} \le 1 - a$ then $x_j = \max\{x_j(1+\beta), \delta\}$

if $\frac{P_j^T y}{C_j^T z} \ge 1 + a$ then $x_j = x_j(1-\beta)$

until convergence

MPCSolver



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$\widetilde{O}\left(\frac{1}{\epsilon^5} \ln^3(kmMnx_{\max})\right)$ rounds

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- Parallelization
    - Shared-memory: straightforward
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- Near-optimality: objective at least $(1 - \epsilon)$OPT
    - compute bounds on objective
        - $\lambda_{\min}$ (only covering)
        - $\lambda_{\max}$ (only packing)
    - add constraint $w^T x \geq \lambda$
    - binary search for λ
    - at most $\log_2 \log_{1-\epsilon} \left(\frac{\lambda_{\min}}{\lambda_{\max}} \right)$ steps

Outline

GBM Rounding

Input: a solution of GBM_ϵ (near-optimal, ϵ-feasible, fractional)
Output: an integral solution that preserves

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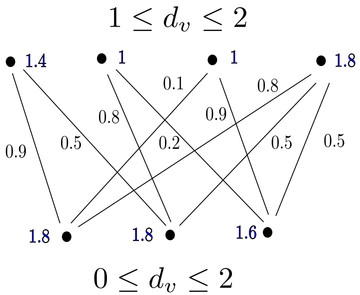

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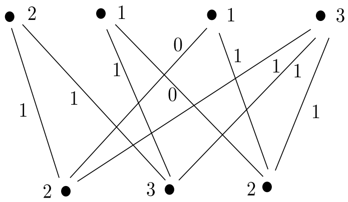

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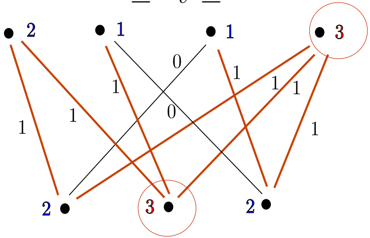

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- Satisfies (2) in expectation
- Violates (1)
- need for dependent rounding

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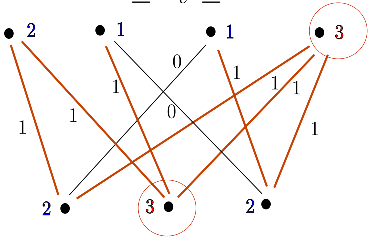

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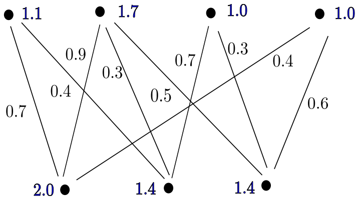
Dependent Rounding

Warm up

Randomized *sequential* Rounding [Gandhi et al. 2006]



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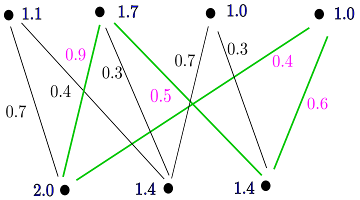

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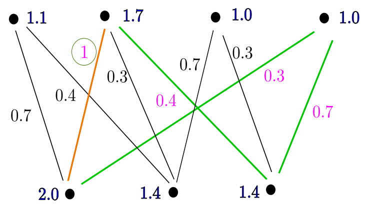

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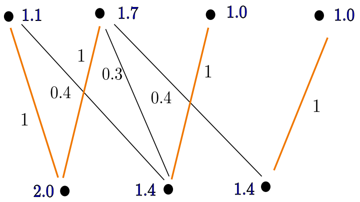

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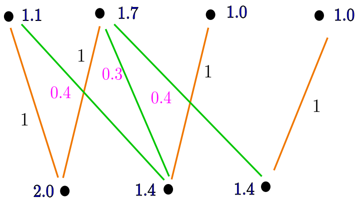

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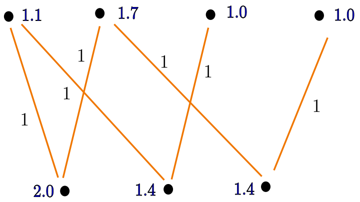
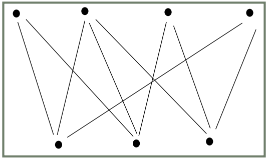


Distributed Rounding

- Partition edges to compute nodes
  - size of each node: number of vertices
  - each node: same number of edges



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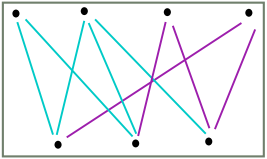

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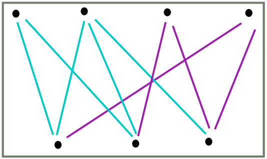
- local cycle $\Rightarrow$ global cycle
- local maximal path $\not\Rightarrow$ global maximal path



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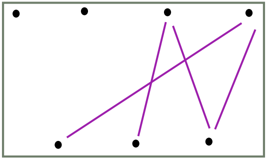

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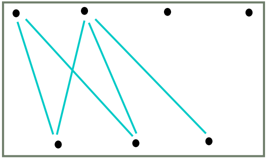

Node 2

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   - k compute nodes, m vertices $\Rightarrow O(mk)$ edges left
3. merge remaining fractional edges (conceptually)
4. Repeat until graph small
5. Run sequential version (cycles, max. paths)

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4. Repeat until graph small
5. Run sequential version (cycles, max. paths)

$$\text{time: } O\left(r^{2+\gamma}\lceil c/\gamma\rceil\right) \begin{cases} n = r^{1+c} & \text{number of edges} \\ \\ \eta = O(r^{1+\gamma}) & \text{size of node} \end{cases}$$

DDRounding in Practice

- Edges already partitioned by MPCSolver
- Empirical: most work done in first iteration
- Scales nicely
- Further communication improvement:
  - halving available nodes at each iteration
  - so, e.g. only odd nodes send their data
- Further time improvement:
  - a node performs cycle detection using DFS
  - when cycle hit, an edge is rounded
  - restart DFS?
  - no! decompose cycle C to paths C_1 and C_2 (reverse)
  - replace DFS stack of C with $\max C_1, C_2$
  - mark node if no cycle found

Outline

References

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Thank you!