On Nash Equilibria for a Network Creation Game

Network Algorithms and Complexity

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MPLA

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- It assumes, however, a central authority constructing the network.
- In practice, networks are built by selfish agents. i.e the internet!
- How bad can this lack of central authority be? (Price of Anarchy)

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- Each player values small distances between him and every other vertex.

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- $Cost(v, \vec{S}) = \alpha |S_v| + \sum_{w \neq v} \delta(v, w)$, where $\delta(v, w)$ is the distance between v and w in $G(\vec{S})$.

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- In the "uncoordinated" case, we need some kind of stable situation (equilibrium).

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- A **transient NE** is a weak NE where there is a sequence of moves that don't change personal cost and lead to a non-NE.

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The price of anarchy ρ is defined as:

$$\rho = \max_{\vec{S} \text{ is a NE}} \frac{Cost(\vec{S})}{Cost(OPT)}$$

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He will strictly gain. It cannot be a NE. □

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We have pairs that are connected. The remainder have distance of at least 2. So,

$$\begin{aligned} Cost(\vec{S}) &\geq \alpha|E| + 2|E| + 2(n(n-1) - |E|) \\ &= 2n(n-1) + (\alpha - 2)|E|. \end{aligned}$$

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This bound is achieved by any graph of diameter at most 2.

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Social optimum is achieved when $|E|$ is maximum, that is $G(\vec{S})$ is a clique.

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Case 3: $\alpha \geq 2$
OPT still a star and a star **is** a NE. However there may be worse NE.

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- (Tree Conjecture) There exists a constant A such that for all $\alpha > A$ all non-transient equilibria are trees.
- Based on that conjecture, the price of anarchy is at most 5.
- However that conjecture is wrong! (Albers et al.)

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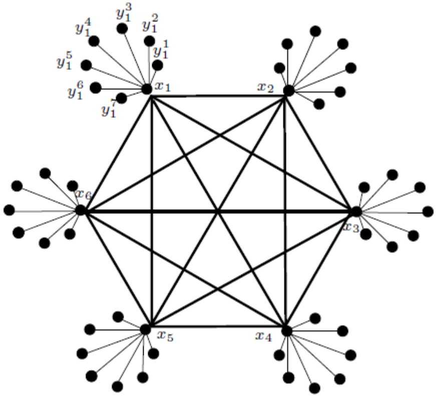
It has been proven that:
For any positive integer n_0, there exists a graph built by $n > n_0$ players that **contains cycles** and forms a strong Nash Equilibrium, for any α, with $1 < \alpha < \sqrt{n/2}$.

Disproving the Tree Conjecture



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- $T(u)$ is not a tree. It is $G(\vec{S})$ layered with distinguished edges.

Improving the bounds for the Price of Anarchy

Types of vertices

Let $G(\vec{S})$ be an NE graph and let $u \in V$. Let $T(u)$ be a shortest path tree rooted at u. We say that a vertex $v \in V$, at a depth smaller than $6 \log n$ in $T(u)$ is:

- **Expanding:** *v has at least two chidren and one descendant in the Boundary level.*
- **Neutral:** *v has exactly one child and at least one descendant in the Boundary level.*
- **Degenerate:** *v has no descendants in the Boundary level.*

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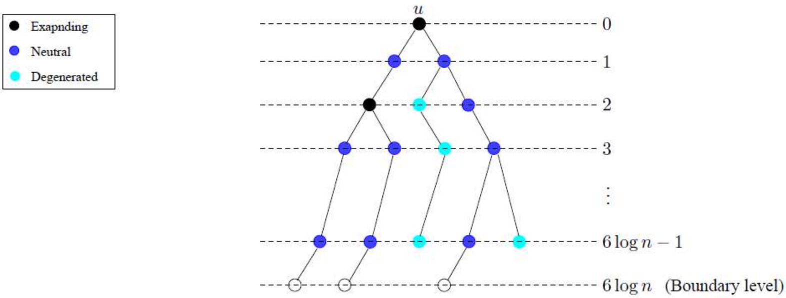

Figure 5: A classification of the vertices of T(u).

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- Given that we have a tree of that depth, we prove the bound.

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Proposition 1

If $G(\vec{S})$ is an equilibrium graph whose girth is at least $12 \log n$ then the diameter of $G(\vec{S})$ is at most $6 \log n$ and $G(\vec{S})$ is a tree.

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We will show that the number of descendants at the Boundary level is at least n, which is a contradiction.

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Proof cont'd:
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Claim

$$N(d, b) \geq 2^{\frac{6 \log n - d}{2} - (2 \log n - b)}$$

That implies that $N(0, 0) \geq n$ which is what we want to prove. □

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This is not an equilibrium graph. Contradiction. □

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$$\frac{\alpha(n-1) + 6n^2 \log n}{\alpha(n-1) + 2(n-1)^2} \leq 1 + \frac{6n^2 \log n}{\alpha n + 2(n-1)^2 - \alpha} \leq 1 + \frac{6n \log n}{\alpha}$$

□

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- All in all, selfish nodes don't behave too bad in this game.
- PoA is bounded for non-trivial values of α. It pays to divide the cases "correctly".

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The end

Thank you!