Coloring Circular Arc Graphs

Revisiting Tucker's algorithm

Alex Angelopoulos

$\mu \prod \lambda \forall$

July 22, 2014

Outline

The problem

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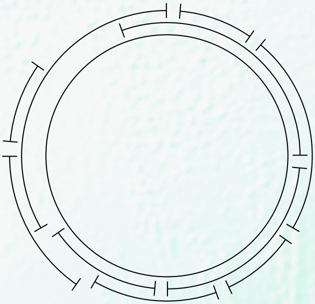
Input: a family F of circular arcs



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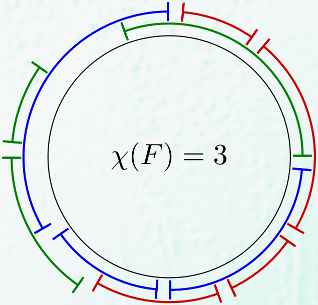
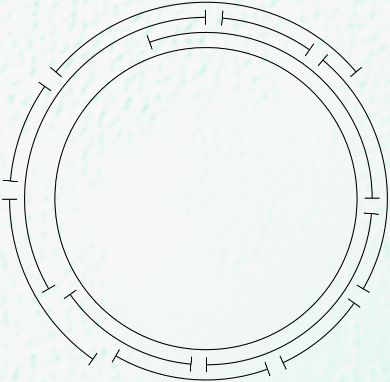


Output: is there a proper coloring with $\leq k$ colors?
what is the minimum k s.t. F has a proper coloring?

Some quantities



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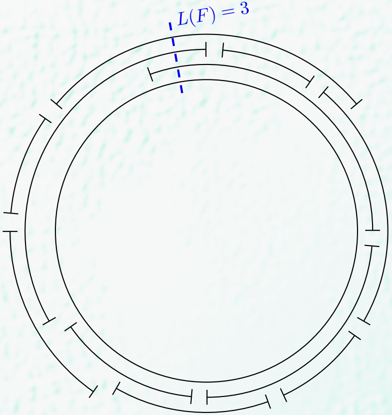
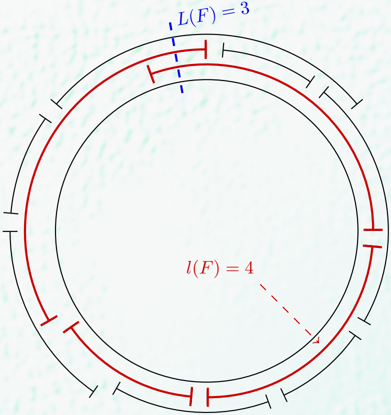


- Load of F: $\boldsymbol{L}$

Some quantities



- Load of F: $\boldsymbol{L}$
- circular-cover: $\boldsymbol{l}$

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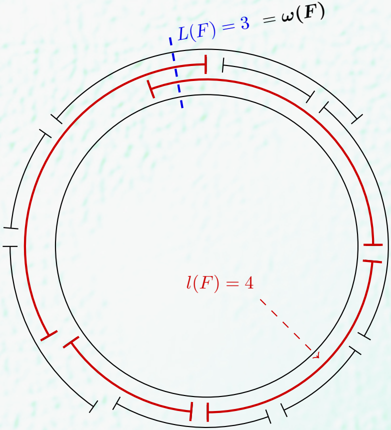

- Load of F: L
- circular-cover: l
- max. clique of F: ω (as usual)

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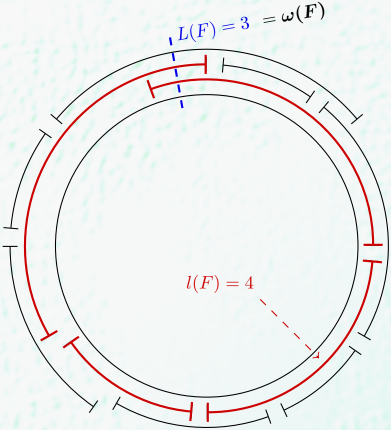

- Load of F: $\boldsymbol{L}$
- circular-cover: $\boldsymbol{l}$
- max. clique of F: $\boldsymbol{\omega}$ (as usual)
- We also discretize and use the -at most- $2|F|$ points defining the arcs.

Results

Theorem 1 (Tucker (1975)).

Let F be a family of circular arcs with load $L = L(F)$ and circular-cover $l = l(F)$. If $l(F) \geq 4$, then $\left\lfloor \frac{3}{2}L \right\rfloor$ colors suffice to properly color F.

- This is actually a **2-approximation** algorithm.
- Tucker, [4] conjecture that $\chi(F) \leq \frac{3}{2}\omega(F)$.
- Karapetian (1980) proves the above.
- Garey et al. (1980) show **NP**-completeness for CIRCULAR ARC COLOR
- Many exact algos for subfamilies of graphs ($\geq O(|F|^{1.5})$).

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More recently:

Theorem 2 (Valencia-Pabon (2003)).

Consider F with load $L(F)$ and circular-cover $l(F) \geq 5$. Then $\left\lceil \frac{l-1}{l-2} L \right\rceil$ colors suffice to color F, bound being tight.

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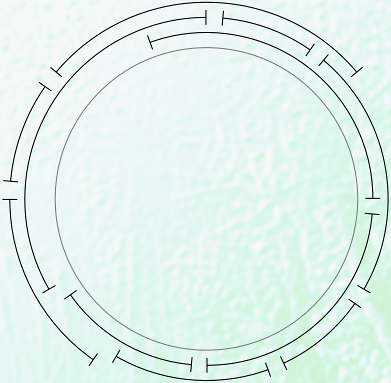
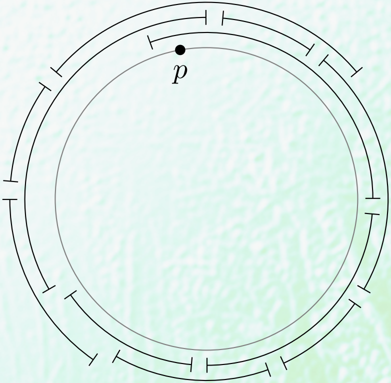
Tucker's algorithm

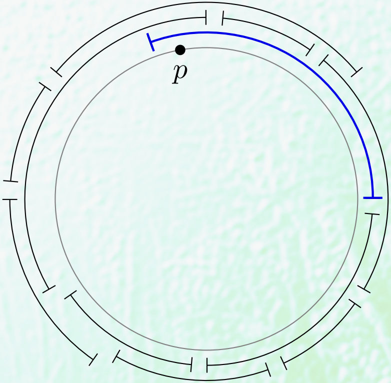


- Select p so that $L(F)$ arcs contain it



Tucker's algorithm

- Select p so that $L(F)$ arcs contain it
- Assign color $\#1$ to the arc which extends at least on the counterclockwise side of p



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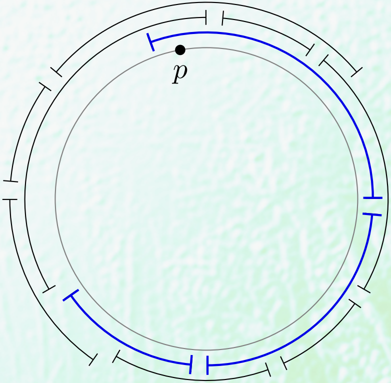

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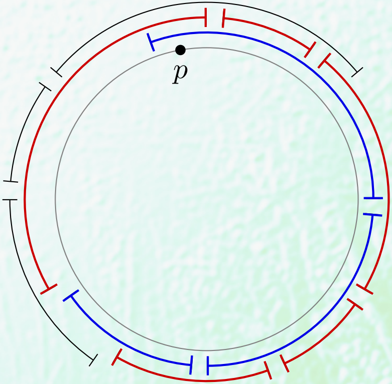

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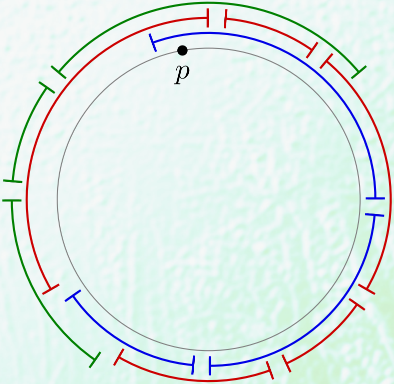
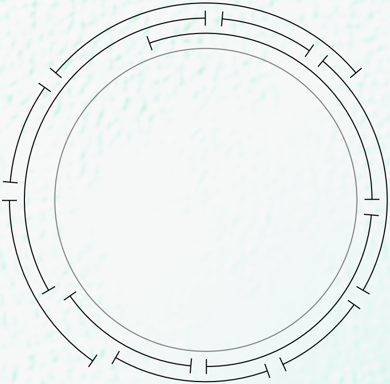


Notations needed



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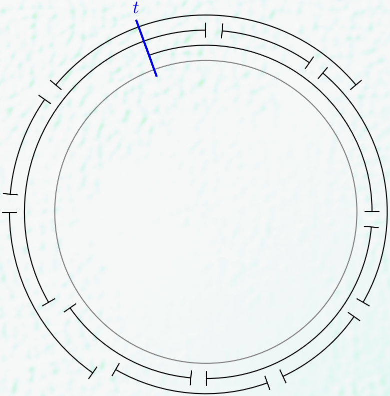

- starting point: t

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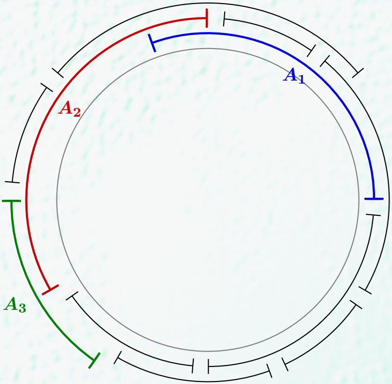

- starting point: t
- first arc colored with i: A_i

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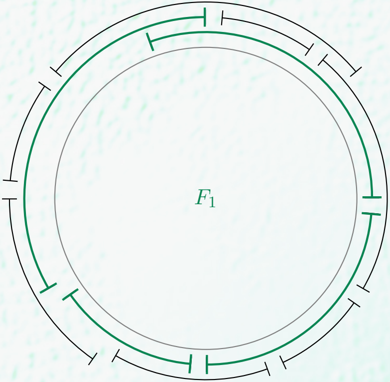

- starting point: $\boldsymbol{t}$
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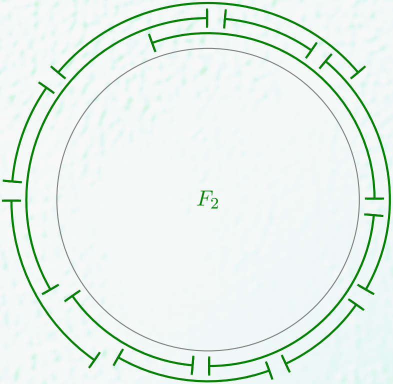

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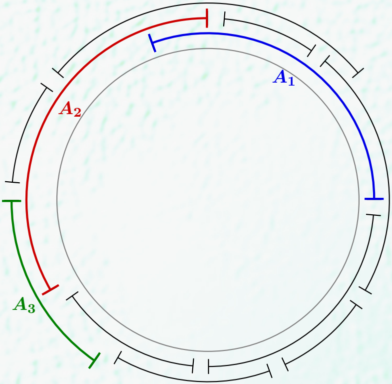
Properties



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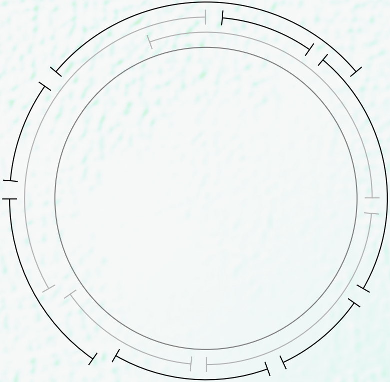

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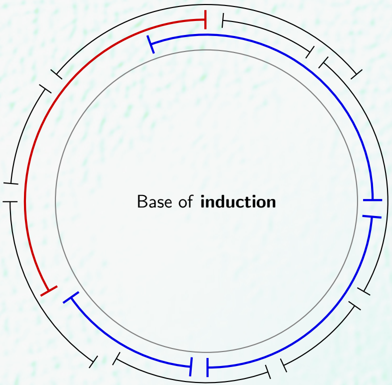

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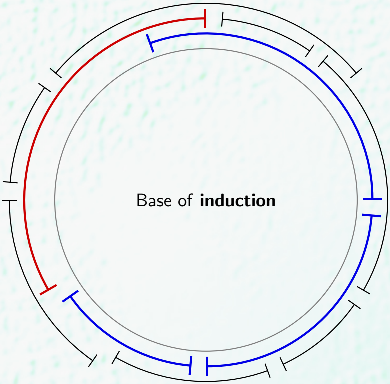

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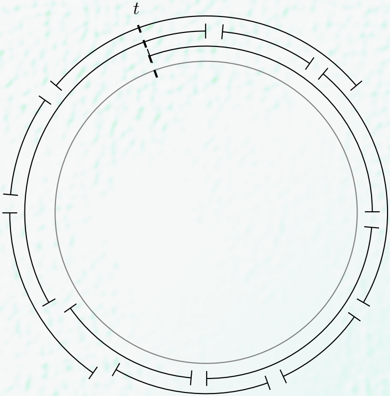
   Proof?

Proving Property 3

$1 \leq i \leq l - 2$



Suppose F_{i-1} is properly colored with i colors. Let:

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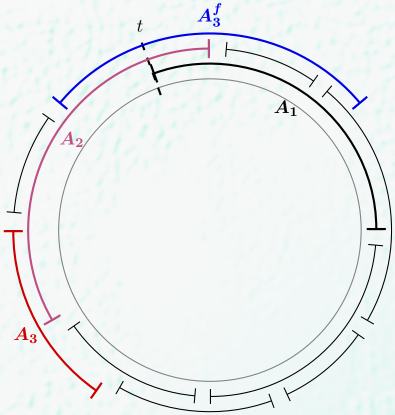

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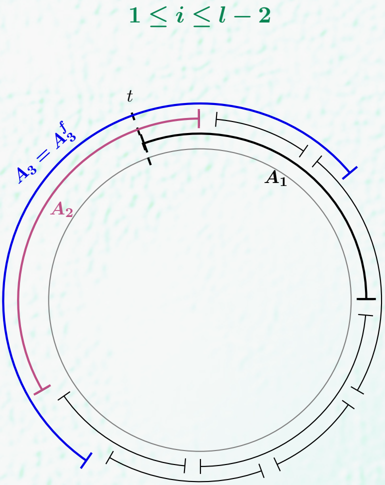

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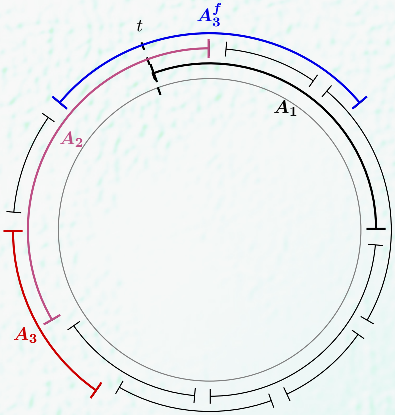

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- Now think about A_{i+1} and A_{i+1}^f

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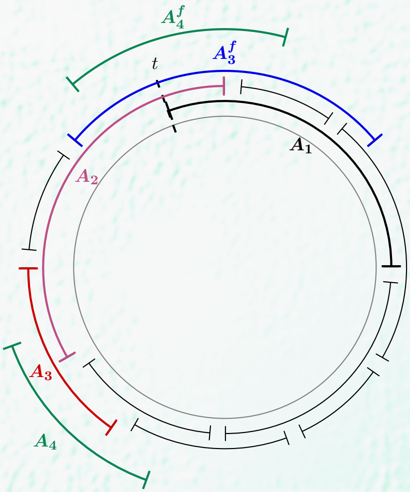

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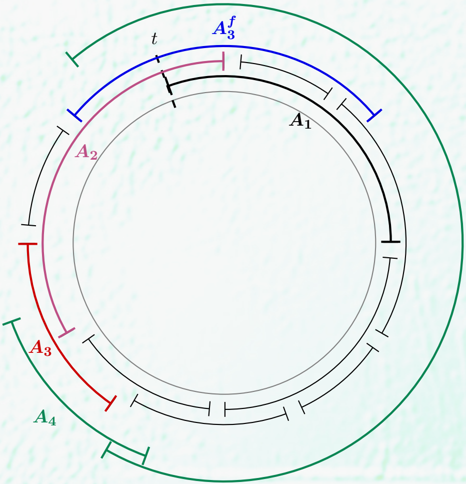

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  In this case: $A_2, A_3, ..., A_i, A_i^f$ **cover** the circle!

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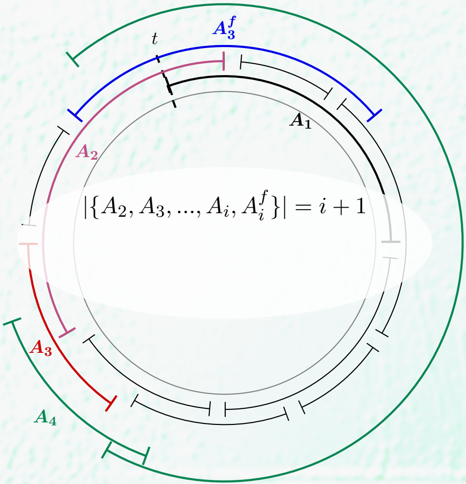
$\mathbf{1 \leq i \leq l - 2}$



Suppose F_{i-1} is properly colored with i colors. Let:

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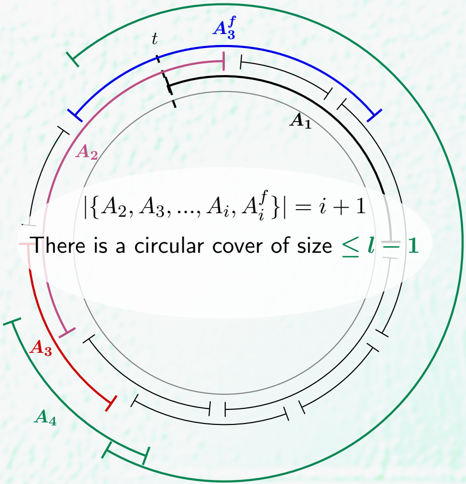

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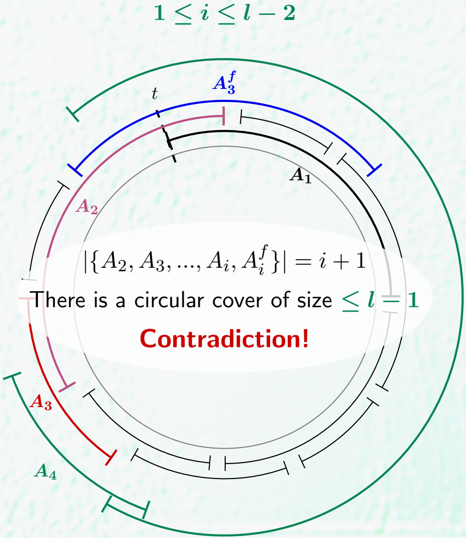

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Recap

Property 2. $L(F \setminus F_{l-2}) \leq L - (l-2)$

Property 3. F_{l-2} are properly colored with $l-1$ colors.

But how to use some induction here?

Maintain little arcs $(p, p+1)$ to create a constant load of $L(F)$ around the circle. Neither $\chi(F)$ nor the Algorithm's output is changed! *Now check again Property 2!

Now: $F' = F \setminus F_{l-2}$ has $l_{F'} \geq l_F \geq \mathbf{5}$, so use induction!...

- L rounds, every $l-2$ rounds need at most $l-1$ colors
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- $\Rightarrow \left\lceil \frac{\boldsymbol{l-1}}{\boldsymbol{l-2}} \boldsymbol{L} \right\rceil$ is the **output** of the algorithm.

Some final interaction

- What if every arc in F spans at most n/k "points" of the circle?

  $l \geq k+1 \Rightarrow SOL = \lceil \frac{k}{k-1} L \rceil$

- What if every arc spans more than a semi-circle?

  The circular arc graph is **complete!**

- To show tightness, Valencia-Pabon uses a result of Stahl, [3] regarding r-tuple colorings...

Major open problem: **better** than $\frac{3}{2}$-approximation?

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  The circular arc graph is **complete!**

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That's all folks!

Thank you!

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