Constrained Matching Problem in Bipartite Graphs

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Definition 1

Given a graph $G = (V, E)$, a matching M in G is a set of pairwise non-adjacent edges; that is, no two edges share a common vertex.

Definition 2 (Bounded Color Matching)

Input is:

- Bipartite graph $G(V, E)$ with bipartition $V = V_1 \cup V_2$.
- The edge set E is partitioned into k sets, $E_1 \cup E_2 \cup \cdots \cup E_k$.
- Each edge set is characterized by a color $j \in [k]$.
- Each edge $e \in E$ has a profit $p_e \in \mathbb{Q}^+$.

Objective is:

- Find a maximum weight matching M.
- In M there are no more that w_j edges of color j where $w_j \in \mathbb{Z}^+$, i.e. $M \cap E_j \leq w_j, \quad \forall j \in [k]$.

LP of Bounded Color Matching

The relaxation of the IP for the Bounded Color Matching problem:

$$\begin{aligned} \text{maximize} \quad & \max p^T x \\ \text{subject to} \quad & \sum_{e \in \delta(v)} x_e \leq 1, \quad \forall v \in V \\ & \sum_{e \in E_j} x_e \leq w_j, \quad \forall j \in [k] \\ & 0 \leq x_e \leq 1 \end{aligned}$$

where $\delta(v)$ is the set of edges with one endpoints in v. Integrality Gap is $\frac{1}{2}$ so we cannot hope to achieve a better that $\frac{1}{2}$ approximation algorithm.

Another way to describe the problem, say $\mathcal{M}'$, is the following:

$$\mathcal{M}' = \left\{ y \in \{0,1\}^{|E|} : y \in \mathcal{M} \wedge \sum_{e \in E_j} y_e \leq w_j, \forall j \in [k] \right\}$$

where $\mathcal{M}$ is the usual bipartite polytope. Again, we can relax this by setting $y_e \in [0,1]$.

- The Bounded Color Matching problem is known to be $\mathbf{NP}-\textit{Complete}$
- Even if $|E_j| \leq 2$ and $w_j = 1, \forall j$.
- The special case of a 2-regular bipartite graphs where,
    1. Each color appears twice.
    2. Find a maximum matching with at most one edge per color

  is $\mathbf{APX}-\textit{Hard}$ so a $\mathbf{PTAS}$ is out of reach.

Definition 3

Let $E' \subseteq E$. Then we define the **characteristic vector** of E' to be the binary vector $\chi_{E'} \in \{0,1\}^{|E|}$, s.t.

$$\chi_{E'}(e) = 1 \Leftrightarrow e \in E'$$

Definition 4

Let $y \in \mathbb{R}^n$. Then,

$$support(y) = \{i \in [n] : y_i \neq 0\}$$

i.e. the indices of the non-zero components of y.

Lemma 5

Let x^* be an optimal basic feasible solution for the LP described by $\mathcal{M}'$ s.t. $x^*_e > 0$, $\forall e \in E$. Then, there exist $F \subseteq V$ and $Q \subseteq [k]$ s.t.,

1. $\sum_{e \in \delta(v)} x^*_e = 1, \quad \forall v \in F.$
2. $\sum_{e \in E_j} x^*_e = w_j, \quad \forall j \in Q.$
3. $\{\chi_{\delta(v)}\}_{v \in F}$ *and* $\{\chi_{E_j}\}_{j \in Q}$ *are all linearly independent.*
4. $|E| = |F| + |Q|$ *where* $|E|$ *is the number of the edges with* $x^*_e > 0$.

- If $\sum_{e \in \delta(v)} x_e = 1$ then v is a *tight vertex*.
- If $\sum_{e \in E_j} x_e = w_j$ then j is a *tight color class*.

Define the **residual graph** to be the graph with the same vertex set but we include an edge e if $x_e > 0$ in the LP solution for the original graph.

Lemma 6

Take any basic feasible solution x s.t. $x_e > 0, \forall e$, i.e. we remove any edge with $x_e = 0$. Then one of the following must be true:

1. *either there is a an edge s.t. $x_e = 1$.*
2. *or there is a color class $j \in Q \subseteq [k]$ s.t. $|E_j| \leq w_j + 1$ in the residual graph*
3. *or there is a tight vertex $v \in F$ s.t. the degree of v is 2 in the residual graph.*

Using the lemma above will can do iterative rounding to obtain a solution.

C, E will be resp. the set of the available colors and edges, at each round.

Initialize $M = \emptyset$.
While $C \neq \emptyset$ or $E \neq \emptyset$ do:

1. Compute an optimal (fractional) basic solution x to the current LP.
2. Remove all edges from the graph s.t. $x_e = 0$.
3. Remove all vertices of the graph s.t. $deg(v) = 0$.
4. if $\exists e = (u, v) \in E: \ x_e = 1$ and $e \in C_j$ then $M := M \cup \{e\}$, $V = V \setminus \{u, v\}$, $w_j := w_j - 1$. if $w_j = 0$ then $C := C \setminus C_j$, $E := E \setminus \{e : e \in E_j\}$.
5. **(Relaxation:)** while $V \cup C \neq \emptyset$
   - if $\exists$ color class $C_j \in Q$ with $|E_j| \leq w_j + 1$ then remove the constraint for this color class, i.e. define $C := C \setminus C_j$.
   - if $\exists$ vertex $v \in F$ s.t. $deg(v) = 2$ then remove the constraint for that vertex.

Return M

At each step of the algorithm, either we add an edge to our matching M, or we remove a tight constraint. Thus the algorithm will terminate in at most $|Q| + |F|$.

- Since we remove the degree constraints for a vertex v when $deg(v) = 2$ we select edges from a graph G' that is a collection of disjoint paths or cycles.
- But a disjoint path or cycle can be partitioned into two matchings, i.e. M_1, M_2 an we select the one with the highest profit, i.e.

$$\max(p(M_1), p(M_2)) \geq \frac{1}{2} p(M_1 \cup M_2)$$

- Therefore we do this for every connected component (disjoints paths and cycles), we get at least $\frac{1}{2}$ of the profit of the matchings but we violate by an additive 1 every color constraint.

As a result of the above there is a polynomial time (1/2, additive 1) bi-criteria approximation algorithm for the weighted Bounded Color Matching problem.

We now consider the unweighted version of the Bounded Color Matching problem: Compute a maximum cardinality matching M s.t. in M we have at most w_j edges for color class j.

Recall that from Lemma 7 we have that for any solution to the LP, if $0 < x_e < 1$ then,

- either there exists a tight color class $j \in Q$ s.t. $|support(x) \cap E_j| \leq w_j + 1$
- or there exists a tight vertex $v \in F$ s.t. $deg(v) = 2$.

The main idea of the algorithm for the cardinality version consists of the two following steps:

- **Relaxation step**: We identify a tight color class j and we remove its constraint, thus relaxing the problem.
- **Rounding step**
  - We round appropriately some variables to 1 and some others to 0, preserving feasibility.
  - Rounding step comes with a parameter $\lambda \in [0, 1]$. Idea is that if we round x_e to 1, we need to update the color bound of this color class.
  - Using λ we update the color bound by any value in $[x_e, 1]$ (if we use $x_e + \lambda(1 - x_e)$).
  - Values of λ closer to x_e violate mode the color constraint whereas values closer to 1 give less violation but worst performance guarantee.

Lemma 7

Let x be the optimal solution in G (as stated in Lemma 4) before the rounding step and $\hat{x}$ be the optimal solution after the rounding step in $\hat{G}$. Then we have that,

$$\sum_{e \in E(G)} x_e - \sum_{e \in E(\hat{G})} \hat{x}_e \leq 1 + (\gamma + \lambda\gamma)$$

where $\gamma = 1 - x_e$.
The loss due to a single rounding step is at most $\gamma + \lambda\gamma$ which can be at most $\frac{1}{2}(\lambda + 1)$.

C, E will be resp. the set of the available colors and edges, at each round.

Initialize $M = \emptyset$.
While $C \neq \emptyset$ or $E \neq \emptyset$ do:

1. Compute an optimal (fractional) basic solution x to the current LP.
2. Remove all edges from the graph s.t. $x_e = 0$.
3. Remove all vertices of the graph s.t. $deg(v) = 0$.
4. if $\exists e = (u, v) \in E:\ x_e = 1$ and $e \in C_j$ then $M := M \cup \{e\}$, $V = V \setminus \{u, v\}$, $w_j := w_j - 1$. if $w_j = 0$ then $C := C \setminus C_j$, $E := E \setminus \{e : e \in E_j\}$.
5. **(Relaxation:)** If $\exists$ color class $j \in Q$ with $|E| \leq \lceil w_j \rceil + 1$ then remove the constraint for this color class, i.e. set $C := C \setminus C_j$ and iterate.
6. **(Rounding:)** if $\exists v \in F$ s.t. $deg(v) = 2$ then let: u_1, u_2 be the neighbors of v and let e_1, e_2 be the two edges incident on v. Assume w.l.o.g. that $x_{e_1} \geq \frac{1}{2}$ and $e_1 = (u_1, v)$.
   - Round x_{e_1} to 1. Add it (e_1) to M.
   - Round x_{e_2} and all other edges incident to u_1 to zero.
   - If $e_1 \in E_j$ then set $w_j := w_j - x_{e_1} - \lambda(1 - x_{e_1})$.
   - Remove v, u_1 and all the rounded edges from the graph and iterate.

Return M

- From Lemma 7 we have that in each rounding step the objective function decreases by $1 + \gamma + \lambda\gamma$.
- Intuitively, the larger the value of γ is, the fewer iterations the algorithm will perform.
- Because $OPT \leq |V|/2$ and at each rounding step we delete 2 vertices from the current graph, we can perform at most $|V|/4$ rounding steps. So, we can have at most $|V|/4$ values of γ, though they all might be different.

Lemma 8

Let $\tilde{x}$ be the final solution of the algorithm that corresponds to M. Then we have that,

$$\sum_{e \in M} \tilde{x}_e \geq \frac{2}{3+\lambda} \sum_{e \in E(G)} x_e$$

Proof.

- Since we choose $x_{e_1} \geq 1/2$ assume that in some iteration $\gamma_1 = \frac{p}{q} \in (0, 1/2]$ and also that this γ_1 appears k_1 times during the Rounding steps.
- The total decrease in the objective function is $\frac{q+p(\lambda+1)}{q} = 1 + \gamma_1 + \lambda\gamma_1$
- Maximum number of iterations we can have for this particular γ_1 is $OPT \cdot \frac{q+p(\lambda+1)}{q}$ before it truncates to 0. E.g. for $\gamma_1 = \frac{1}{3}$ and $\lambda = \frac{1}{2}$ in the next iteration of the LP we will have

$$OPT' = OPT - \frac{3}{2} \Rightarrow OPT - OPT' = \frac{3}{2}$$

  and so we can have at most $OPT \cdot \frac{2}{3}$ iterations.

□

continued.

- Assume that the algorithm performs a fraction f_i of the maximum possible number of iterations for each γ_i. Then we have that,

$$\sum_i f_i \leq 1$$

because in each round we reduce the objective function.

- At the end of the algorithm the final objective function value will be,

$$OPT - \sum_i f_i \cdot \frac{OPT}{1 + \gamma_i + \lambda\gamma_i} \cdot \gamma_i(\lambda + 1)$$

□

continued.

Set $g(\gamma_i) = \frac{\gamma_i(\lambda+1)}{1+\gamma_i+\lambda\gamma_i}$ which monotonically increases. We have that

$$
\begin{aligned}
SOL = OPT - OPT\sum_i f_i \cdot g(\gamma_i) &\geq OPT - OPT\sum_i f_i \cdot g(1/2) \\
&= OPT - OPT\sum_i \frac{\lambda+1}{\lambda+3} \\
&\geq \frac{2}{\lambda+3} OPT
\end{aligned}
$$

Using similar arguments one can show that the color bound w_j of a color j can be violated by at most a factor of $\frac{2}{1+\lambda} w_j + 1$. □

Theorem 9

For any $\lambda \in [0,1]$, there is a polynomial time $(\frac{2}{3+\lambda}, \frac{2}{1+\lambda} w_j + 1)$ bi-criteria approximation algorithm for the Bounded Color Matching problem.

- The closer λ is to 1 the more we deteriorate from the optimal objective function value but the less we lose in color bounds.
- The closer λ is to 0 the more we violate the color constraints but the better the approximation guarantee is.
- Depending on the application we choose a parameter λ that is more suitable.
- We have a family of algorithms for the unweighted case.

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Constrained matching problems in bipartite graphs.
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