Shortest Vector Problem

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May 24, 2011

Contents

Lattice

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Definition

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$$\mathcal{L} = \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{a_i} \mid \lambda_i \in \mathbb{Z} \right\}$$

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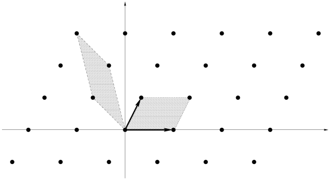

Figure: A Lattice in $\Re^2$

More useful things

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Theorem

Let $\mathcal{L}$ be a n-dimensional lattice and

- $\mathbf{A}$ *be the $n \times n$ matrix whose rows are the basis* $\mathbf{a_1}, \cdots, \mathbf{a_n}$.
- $\mathbf{B}$ *be the $n \times n$ matrix whose rows* $\mathbf{b_1}, \cdots, \mathbf{b_n} \in \mathcal{L}$.

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1. $\mathbf{b_1, \cdots, b_n}$ *form a basis for $\mathcal{L}$.*
2. $|\det(\mathbf{A})| = |\det(\mathbf{B})|$.
3. *there is an $n \times n$ matrix* $\mathbf{U}$ *such that* $\mathbf{B = UA}$ *and* $|\det(\mathbf{U})| = 1$.

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3. *there is an $n \times n$ matrix* $\mathbf{U}$ *such that* $\mathbf{B} = \mathbf{UA}$ *and* $|\det(\mathbf{U})| = 1$.

So the determinant of all the bases of $\mathcal{L}$ is invariable.
$\det(\mathbf{A}) = \det \mathcal{L}$

More useful things

Definition

The *Eucledian norm* of a vector $\mathbf{x}$ is $\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}$ where $x_1, x_2, \cdots, x_n$ are the coefficients in an orthonormal system.

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Orthogonality defect of the basis $\mathbf{a_1}, \mathbf{a_2}, \cdots, \mathbf{a_n}$

$$\frac{\|\mathbf{a_1}\| \cdots \|\mathbf{a_n}\|}{\det \mathcal{L}}$$

Primitivity

Definition

The linearly independent vectors $\mathbf{b_1}, \cdots, \mathbf{b_k} \in \mathcal{L}$ are *primitive* if they can be extended to a basis of $\mathcal{L}$.

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Definition

A vector $\mathbf{a} \in \mathcal{L}$ is *shortest in its direction* if $x\mathbf{a}$ is not in $\mathcal{L}$ for $0 < x < 1$.

Theorem

Vector $\mathbf{a} \in \mathcal{L}$ is primitive iff $\mathbf{a}$ is shortest in its direction.

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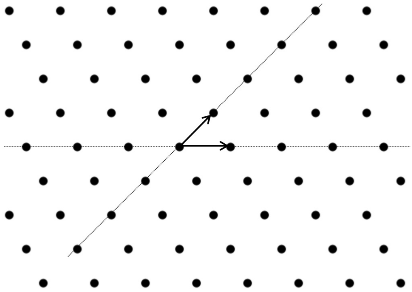
Given a lattice $\mathcal{L}$, find the shortest vector, in Eucledian norm, in $\mathcal{L}$.



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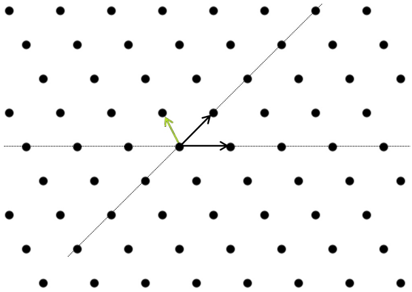

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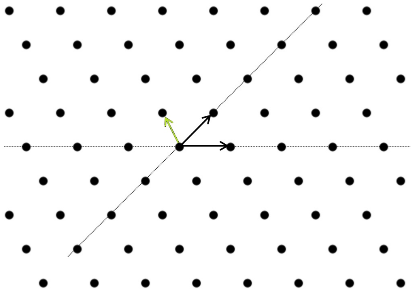

The shortest of the basis is not always the shortest vector.

Contents

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Step 1 is called *Lattice (Basis) Reduction*

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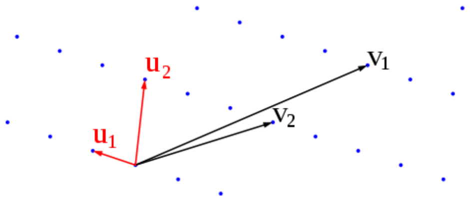

Algorithm for the 1-dimensional Lattices
Euclid's Algorithm

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  Then the shortest vector is the smallest number expressed as an integer l.c. of a, b: $\gcd(a, b)$.
  Euclid's algorithm: $\gcd(a, b) = gcd(b, a - mb)$ where m is the integer closest to a/b.

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Gauss reduced basis

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Theorem

Suppose $\mathbf{b_1}, \mathbf{b_2}$ *is a basis for a 2-d* $\mathcal{L}$ *and* $\|\mathbf{b_1}\| \leq \|\mathbf{b_2}\|$*. Suppose* $\theta \in (0^\circ, 180^\circ)$ *the angle between the two vectors. If* $60^\circ \leq \theta \leq 120^\circ$ *then* $\mathbf{b_1}$ *is the shortest vector in* $\mathcal{L}$*.*

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Let $\mu_{21}\mathbf{b_1}$ denote the projection of the vector $\mathbf{b_2}$ in the direction of the vector $\mathbf{b_1}$:

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If $\|\mathbf{b_1}\| \leq \|\mathbf{b_2}\|$ and $|\mu_{21}| \leq 1/2$ then the basis $\mathbf{b_1}, \mathbf{b_2}$ is Gauss reduced basis.

Gauss's Algorithm

Algorithm for S.V. in 2-d

repeat
  if $\|\mathbf{b_1}\| > \|\mathbf{b_2}\|$ **then**
    swap $\mathbf{b_1}, \mathbf{b_2}$
  end if
  $m \leftarrow \lfloor \mu_{21} \rceil$
  $\mathbf{b_2} \leftarrow \mathbf{b_2} - m\mathbf{b_1}$
until $\|\mathbf{b_1}\| < \|\mathbf{b_2}\|$
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- Polynomial complexity

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$$\mathbf{b_1^*} = \mathbf{b_1}$$

$$\mathbf{b_i^*} = \mathbf{b_i} - \sum_{j=1}^{i-1} \frac{\mathbf{b_i b_j^*}}{\|\mathbf{b_j^*}\|^2} \mathbf{b_j^*}$$

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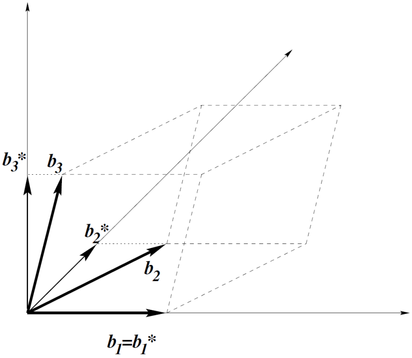
Define

$$\mu_{ij} = \frac{\mathbf{b_i b_j^*}}{\|\mathbf{b_j^*}\|^2}$$

and $\mu_{ii} = 1$. Then

$$\mathbf{b_i} = \sum_{j=1}^{i-1} \mu_{ij} \mathbf{b_j^*}$$

Lower bounding OPT



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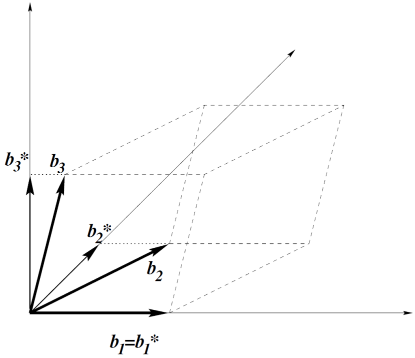

If OPT is the length of the shortest vector in the lattice then:

$$OPT \geq min\{\|\mathbf{b_1^*}\|, \cdots, \|\mathbf{b_n^*}\|\}$$

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An LLL Reduced Basis is reasonably orthogonal:

$$\frac{\|\mathbf{b_1}\| \cdots \|\mathbf{b_n}\|}{\det \mathcal{L}} \leq 2^{\frac{n(n-1)}{2}}$$

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In an LLL Reduced Basis we have that:

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In an LLL Reduced Basis we have that:

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So we want an algorithm which turns an arbitrary basis into an LLL reduced in polynomial time in n.

Contents

Shortest vector Algorithm

LLL $(\mathbf{B} = \mathbf{b_1}, \cdots, \mathbf{b_n})$

  $\mathbf{B} \leftarrow \mathit{SizeReduce}(\mathbf{B})$

  while $\exists i$ violating $\|\mathbf{b_i^*}\|^2 \leq \frac{4}{3}\|\mathbf{b_{i+1}^*} + \mu_{i+1,i}\mathbf{b_i^*}\|^2$ **do**

    swap $\mathbf{b_i}, \mathbf{b_{i+1}}$

    update μ_{hk} and $\mathbf{b_k^*}$ for all h, k

    $\mathbf{B} \leftarrow \mathit{SizeReduce}(\mathbf{B})$

  end while

  return $\mathbf{b_1}$

SizeReduce $(\mathbf{B} = \mathbf{b_1}, \cdots, \mathbf{b_n})$

  for $j = 2, \cdots, n$ **do**

    for $i = j - 1, \cdots, 1$ **do**

      $\mathbf{b_j} \leftarrow \mathbf{b_j} - \lfloor \mu_{ji}\mathbf{b_i} \rceil$

      $\mu_{jk} \leftarrow \mu_{jk} - \mu_{ji}\mathbf{b_i}\mu_{ik}$ for $k = 1, \cdots, i$

    end for

  end for

  return $\mathbf{B}$

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The *dual lattice* $\mathcal{L}^*$ of the lattice $\mathcal{L}$ is defined by:

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Theorem

Let $\mathbf{b_1} \cdots \mathbf{b_n}$ be any basis for $\mathcal{L}$. Then, the rows of $\mathbf{B^{-T}}$ form a basis for the dual lattice $\mathcal{L}^*$. Furthermore, $\det \mathcal{L}^* = \frac{1}{\det \mathcal{L}}$.

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Theorem

Let $\mathbf{b_1} \cdots \mathbf{b_n}$ *be any basis for* $\mathcal{L}$*. Then, the rows of* $\mathbf{B}^{-\mathbf{T}}$ *form a basis for the dual lattice* $\mathcal{L}^*$*. Furthermore,* $\det \mathcal{L}^* = \frac{1}{\det \mathcal{L}}$.

Definition

Let $\boldsymbol{v} \in \Re^n$ be a non-zero vector. Then, $\boldsymbol{v}^\perp$ will denote the $(n-1)$-dimensional space $\{\mathbf{b} \in \Re^n | \mathbf{b} \cdot \boldsymbol{v} = 0\}$

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Lemma

Let $\boldsymbol{v} \in \mathcal{L}^*$ be primitive. Then

- $\mathcal{L} \cap (\boldsymbol{v}^\perp)$ *is an* $(n-1)$*-dimensional sublattice of* $\mathcal{L}$.
- *There is a vector* $\mathbf{b} \in \mathcal{L}$ *such that* $\boldsymbol{v} \cdot \mathbf{b} = 1$

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Lemma

We can create a basis $\mathbf{w}_n, \cdots, \mathbf{w}_1$ *with Gram-Schmidt orthogonalization* $\left(\frac{\boldsymbol{v}_n}{\|\boldsymbol{v}_n\|^2}, \cdots, \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|^2}\right)$.

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Gram-Schmidt lower bound is not so bad

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Minkowski's theorem

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There is a basis for $\mathcal{L}$ whose Gram-Schmidt lower bound is at least OPT/n.

Contents

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If f is a polynomial modulo n and h is a polynomial having the same roots as f modulo n and has 'small' norm then all the roots of f (smaller than some value) are also roots of h over the integers.

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The LLL algorithm can find such a polynomial h and then solve the equation $h(x) = 0$ over the integers to get small solutions.

QUESTIONS