Facility Location[1]

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Outline

Definition

Let G be a bipartite graph with bipartition (F, C), where F is the set of *facilities* and C is the set of *cities*. Let f_i be the cost of opening facility i, and c_{ij} be the cost of connecting city j to open facility i. The connection costs satisfy the triangle inequality. The problem is to find a subset $I \subseteq F$ of facilities that should be opened, and a function $\phi : C \longrightarrow I$ assigning cities to open facilities in such a way that the total cost of opening facilities and connecties cities to open facilities is minimized.

Problem Formulation of the Primal

$$\min \sum_{i\in F, j\in C} c_{ij}x_{ij} + \sum_{i\in F} f_i y_i \quad (1)$$

$$\sum_{i\in F} x_{ij} \geq 1, \quad j \in C \quad (2)$$

$$y_i - x_{ij} \geq 0, \quad i \in F, j \in C \quad (3)$$

$$x_{ij} \in \{0,1\}, \quad i \in F, j \in C \quad (4)$$

$$y_i \in \{0,1\}, \quad i \in F \quad (5)$$

Constraint (2) ensures that each city j is connected to at least one facility, and constraint (3) ensures that this facility must be open.

The Dual Problem

$$\max \sum_{j \in C} a_j \tag{6}$$

$$\alpha_i - \beta_{ij} \leq c_{ij}, \quad i \in F, j \in C \tag{7}$$

$$\sum_{j \in C} \beta_{ij} \leq f_i, \quad i \in F \tag{8}$$

$$\alpha_j \geq 0, \quad j \in C \tag{9}$$

$$\beta_{ij} \geq 0, \quad i \in F, j \in C \tag{10}$$

An Intuitive Understanding of the Dual

$$\forall i \in F, j \in C : x_{ij} > 0 \Rightarrow \alpha_j - \beta_{ij} = c_{ij} \tag{11}$$

$$\forall i \in F : y_i > 0 \Rightarrow \sum_{j \in C} \beta_{ij} = f_i \tag{12}$$

$$\forall j \in C : \alpha_j > 0 \Rightarrow \sum_{i \in F} x_{ij} = 1 \tag{13}$$

$$\forall i \in F, j \in C : \beta_{ij} > 0 \Rightarrow y_i = x_{ij} \tag{14}$$

By condition (14), if facility i is open, but $\phi(j) \neq i$, then $y_i \neq x_{ij}$, and so $\beta_{ij} = 0$. By (12), each open facility must be fully paid. By (11), we can think of α_j as the total price paid by city j, of this c_{ij} goes towards the use of edge (i, j), and β_{ij} is the contribution of j towards opening facility i.

Algorithm

Phase 1

Algorithm 1 Phase 1

```
while (j=next_unconnected_city(mod(j+1,|C|)))> 0 do
  α_j = α_j + 1
  for i = 1 to |F| do
    if α_j == c_ij then
      set_tight ((i,j))
      if α_j − β_ij ≤ c_ij then
        β_ij = β_ij + 1
        set_special ((i,j))
      end if
    end if
    if Σ_j β_ij == f_i then
      set_temporarily_open (facility_i)
    end if
    if city_j.unconnected() and is_tight((i,j)) and facility_i.is_temporarily_open() then
      facility_i.set_connected_witness (city_j)
      city_j.set_connected ()
      β_ij = 0
    end if
  end for
end while
```

Algorithm

Phase 2

Let F_t denote the set of temporarily open facilities. Let $T \subset G$, consisting of all special edges. Let $H \subseteq T^2$ induced on F_t. We need any maximal independent set in I in H.

Algorithm 2 Phase 2

```
for j = 1 to |C| do
    F_j = {i ∈ F_t : (i, j) is special}
    if facility_i ∈ (F_j ∩ I) then
        φ(j) = i
        city_j = set_directly_connected ()
    else if facility_i'.is_connecting_witness (j) and is_tight ((i', j)) then
        if i' ∈ I then
            φ(j) = i'
            city_j = set_directly_connected ()
        else
            i'' = i'.get_neighbor ()
            φ(j) = i''
            city_j = set_indirectly_connected ()
        end if
    end if
end for
```

Under the Scope

The Events of Phase 1

Sorting all edges by increasing cost, gives the order and the times at which edges go tight. For each facility i, we maintain the number of cities that are currently contributing towards it, and the anticipated time t_i at which it would be completely paid for if no other event happens on the way (min heap). Running time $O(m \log m)$, where $m = |C||F|$

An edge (i, j) goes tight.

- If facility i is not temporarily open, then it gets one more city contributing towards its cost.
- If facility i is temporarily open, city j is declared connected, and α_j is not raised anymore. For each facility i' that was counting j as a contributor, we need to decrease the number of contributors by 1 and recompute the anticipated time at which it gets paid for.

Under the Scope

The Events of Phase 2

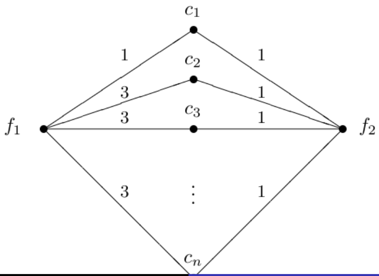
Facility i is completely paid for.

- Facility$_i$ will be declared temporarily open
- All cities contributing to i will be declared connected.
- For each of these cities, we need to decrease by 1 all other facilities that they were contributing towards.

A Tight Example

Problem Instance

The graph has n cities, $c_1, \cdots, c_n$ and two facilities f_1, f_2. Each city is at distance 1 from f_2. City c_1 is at a distance 1 from f_1, and $c_2, , c_n$ are at a distance of 3 from f_1. The opening cost of f_1 and f_2 are ϵ and $(n+1)\epsilon$, respectively, for a small number ϵ.



Optimal Solution - OPT

The optimal solution is to open f_2 and connect all cities to it, at a total cost of $(n+1)\epsilon + n$.

Algorithmic Output Solution - SOL

The algorithm will open f_1 and connect all cities to it, at a total cost of $\epsilon + 1 + 3(n-1)$. Facility f_1 gets fully paid very quickly!

Fault Tolerant Metric Uncapacitated Facility Location

Formulation and Differantiation

Consider a fault tolerant version of the MUFL problem in which every city j is required to be connected to r_j facilities. The problem is to find $I \subseteq F$ of facilities that should be opened, and a function $\phi : C \rightarrow 2^I$ assigning cities to a set of open facilities in such a way that each city j is assigned to a set of cardinality r_j and the total cost of opening facilities and connecting cities to them is minimized.

Fault Tolerant Metric Uncapacitated Facility Location

Problem Formulation of the Primal

$$\min \sum_{i \in F, j \in C} c_{ij} x_{ij} + \sum_{i \in F} f_i y_i \tag{15}$$

$$\sum_{i \in F} x_{ij} \geq r_{ij}, \quad j \in C \tag{16}$$

$$y_i - x_{ij} \geq 0, \quad i \in F, j \in C \tag{17}$$

$$x_{ij} \geq 0, \quad i \in F, j \in C \tag{18}$$

$$1 \geq y_i \geq 0, \quad i \in F \tag{19}$$

Constraint (16) ensures that each city j is connected to at least r_j facilities.

The Dual Problem

$$\max \sum_{j \in C} r_j \alpha_j - \sum_{i \in F} z_i \tag{20}$$

$$\alpha_j - \beta_{ij} \leq c_{ij}, \quad i \in F, j \in C \tag{21}$$

$$\sum_{j \in C} \beta_{ij} \leq f_i + z_i, \quad i \in F \tag{22}$$

$$\alpha_j \geq 0, \quad j \in C \tag{23}$$

$$\beta_{ij} \geq 0, \quad i \in F, j \in C \tag{24}$$

The High Level Algorithm

Principle

Each phase decreases by 1 the maximum *residual requirement*, which is the maximum number of further facilities needed by a city.

Theorem

Cost of (I_{p-1}, C_{p-1}) *minus the cost of* (I_p, C_p) *is at most* $\frac{3OPT}{p}$

Corollary

Cost of (I_0, C_0) *is at most* $3H_k OPT$

The p-th Phase

p-th Phase

Each city j with *residual requirement* of p, for the solution (I_p, C_p) gets connected to at least one more open facility, by either connecting to an already open facility (*free*), or by opening a new one (*priced*).

Problem Formulation of the Primal

Redefining the Primal for the p-th Phase

For phase p The set of free facilities is I_p and the set of cities with residual requirement of p is C_p

$$\min \sum_{i\in F, j\in C_p} c_{ij}x_{ij} + \sum_{i\in F-I_p} f_i y_i \tag{25}$$

$$\sum_{i\in F} x_{ij} \geq 1, \quad j \in C_p \tag{26}$$

$$y_i - x_{ij} \geq 0, \quad i \in F - I_p, j \in C_p \tag{27}$$

$$x_{ij} \geq 0, \quad i \in F, j \in C \tag{28}$$

$$y_i \geq 0, \quad i \in F \tag{29}$$

Redefining the Dual for the p-th Phase

- Duals of only those cities which have residual requirement of p will be raised.
- Connections already used in (I_p, C_p) are of infinite costs.

$$\max \sum_{j \in C_p} a_j \tag{30}$$

$$\alpha_i - \beta_{ij} \leq c_{ij}, \quad i \in F - I_p, j \in C_p \tag{31}$$

$$\alpha_j \leq c_{ij}, \quad i \in I_p, j \in C_p \tag{32}$$

$$\sum_{j \in C} \beta_{ij} \leq f_i, \quad i \in F - I_p \tag{33}$$

$$\alpha_j \geq 0, \quad j \in C \tag{34}$$

$$\beta_{ij} \geq 0, \quad i \in F, j \in C \tag{35}$$

Performance Gap

- City$_j$ is connected to tentatively open facility$_i$, which got closed. Hence, $\alpha_j \geq t_i$ and $\alpha_j \geq c_{ij}$.
- There is a city$_{j'}$ which was paying to i and the open i'. Hence, $\alpha_{j'} \geq c_{ij'}$ and $\alpha_{j'} \geq c_{i'j'}$.
- $\alpha_{j'}$ stopped being raised as soon as one of the facilities i and i' is tentatively open. Hence $\alpha_j \leq \min(t_i, t_{i'})$
- City$_j$ will be connected to the facility$_{i'}$. Thus, $\alpha_{j'} \leq \alpha_j$ and $c_{ij} + c_{ij'} + c_{i'j'} \leq 3\alpha_j$.
- By triangle inequality $c_{i'j} \leq 3\alpha_j$.