The Relative Complexity of Approximate Counting Problems

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Outline

- Definitions
  - Approximate Counting and FPRAS
  - Approximation Preserving (AP) and Parsimonious reductions
  - Fundamental Problems (#SAT and #BIS)
- Classes
  - Problems that admit an FPRAS
  - Problems AP-interreducible with #BIS
  - Problems AP-interreducible with #SAT
- Problems of intermediate Complexity

Approximate Counting

- #P: Counting versions of problems in NP
- FPRAS: A probabilistic Turing machine with input (x, ϵ) approximating a function $f(x)$ has output Y s.t. $Pr(f(x)e^{-\epsilon} \leq Y \leq f(x)e^{\epsilon}) \geq \frac{3}{4}$ and running time polynomial in $|x|$,ϵ^{-1}.

AP reductions

- Reduction of A to B: a TM with a RAS oracle for B which runs in time $poly(|x|, \epsilon^{-1})$ and only asks for an error bound $poly(|x|, \epsilon^{-1})$
- Parsimonious Reduction: a reduction which preserves the number of solutions
- AP-reduction are very liberal, Parsimonious reductions are very strict.

#SAT and #BIS

- #SAT: Compute the number of satisfying assignments to a CNF formula
- #BIS: Compute the number of independent sets in a bipartite graph
- #SAT is #P-complete with respect to AP-reducibility because Cook's theorem uses parsimonious reductions. It admits no FPRAS unless NP=RP

Counting version of NP-COMPLETE pr

- A∈NP-COMPLETE → #A∈#P-COMPLETE (AP)
- Proof:
  - We need to show that #SAT$\leq_{AP}$#A
  - #SAT admits an FPRAS if we have an oracle for SAT (Valiant and Vazirani)
  - An oracle for SAT can be replaced by a RAS for #A

#LargeIS and #IS (i)

- #LargeIS: Given an integer m and a graph whose maximum idependent sets are of size m, how many maximum independent sets are there?
- #LargeIS $\equiv_{AP}$ #SAT because the decision version is NP-complete
- #IS: Given a graph compute the number of independent sets (of any size)
- #LargeIS $\equiv_{AP}$ #IS

#LARGEIS and #IS (ii)

Proof: (boosting technique)

- Construct a new graph from $G = (V, E)$, $V' = V \times [r]$, $E' = \{\{(u,i),(v,j)\} : \{u,v\} \in E \wedge i,j \in [r]\}$
- Independent sets in G' project naturally to independent sets in G
- $(2^r - 1)^m$ different i.s. in G' project to the same size-m i.s. in G
- Thus $|I(G')| \geq (2^r - 1)^m |I_m(G)|$
- i.s. projecting to i.s. of size less than m are at most $(2^r - 1)^{m-1} 2^n$, thus

  $|I(G')| \leq (2^r - 1)^m |I_m(G)|$ + $(2^r - 1)^{m-1} 2^n$
- $\rightarrow |I_m(G)| = \lfloor \frac{|I(G)|}{(2^r - 1)^m} \rfloor$

H-colorings

- An H-coloring of a graph G is a homomorphism from G to H such that adjacent vertices are mapped to adjacent vertices (H may contain self loops).
- Examples: K_q-colorings $\equiv$ normal q-colorings, K_2^1-colorings $\equiv$ independent sets
- #q-PARTICLE-WR-CONFIGS: The number of S_q^*-colorings of a graph G, where S_q^* is the q-leaf star with loops on every vertex

Problems interreducible with #BIS

- #P_4-COL: The number of P_4-colorings where P_4 is a path of 4 nodes
- #DOWNSETS: The number of downsets in a partial order
- #1P1NSAT: The number of satisfying assignments of a restricted CNF formula
- #BEACHCONFIGS: The number of P_4^*-colorings of a graph G

Easy reductions

- #BIS$\equiv_{AP}$#P_4-COL: Nodes with colors 1 and 4 form an independent set.
- #DOWNSETS$\equiv_{AP}$#1P1NSAT: #DOWNSETS is #1P1NSAT without cycles and single-variable clauses

#BIS$\leq_{AP}$ #2-PARTICLE-WR-CONFIG

- Given $B = (X, Y, A)$, let U_i, $0 \leq i \leq n-1$, K be disjoint sets of size $3n$.
- $V' = \bigcup_{i \in [n]} U_i \cup \{v_0, \ldots, v_{n-1}\} \cup K$,
  $E' = \bigcup_{i \in [n]} U_i^{(2)} \cup (v_0, \ldots, v_{n-1} \times K) \cup K^{(2)} \cup \bigcup \{U_i \times \{v_j\} : (x_i, y_j) \in A\}$
- full WR configuration: all cliques are bichromatic
- Colors: red,green, white. Suppose $C(K) = (r, w)$. Project colorings to independent sets: $I = \{x_i : g \in C(U_i)\} \cup \{y_j : C(v_j) = r\}$
- $2(2^{3n} - 2)^{n+1}$ ways
- non-full configurations $\leq 3(n+1)(2 \cdot 2^{3n})^n 3^n$
- $|I(B)| = \lfloor \frac{|W(G)|}{2(2^{3n}-2)^{n+1}} \rfloor$

#2-P-WR-C $\leq_{AP}$ #BEACHCONFIGS

- $V' = V \cup \{s\} \cup [r]$,
  $E' = E \cup (V \times \{s\}) \cup (\{s\} \times [r])$
- $|B(G')| = 2 \cdot 3^r \cdot |W(G)| + 2 \cdot 2^{n+r}$
- Thus $|W(G) = \lfloor \frac{|B(G')|}{2 \cdot 3^r} \rfloor$

#BeachConfigs $\leq_{AP}$ #Downsets

- Construct a partial order on the $3n$ elements of $V \times [3]$.
- For each vertex v, $(v, 0) \prec (v, 1) \prec (v, 2)$.
- For each edge (u, v), $(u, 0) \prec (v, 1)$, $(u, 1) \prec (v, 2)$, $(v, 0) \prec (u, 1)$, $(v, 1) \prec (u, 2)$
- Given a downset D color vertex v with the size of $D \cap \{(v, 0), (v, 1), (v, 2)\}$

#DOWNSETS $\leq_{AP}$ #BIS

- Let $(X, \preceq)$ be an instance of #DOWNSETS.
  Define a bipartite graph $B(U, V, E)$.
  $U = \bigcup_{i \in X} U_i$, $V = \bigcup_{i \in X} V_i$, where
  $|U_i| = |V_i| = 2n$.
  $E = \{(u, v) : u \in U_i \wedge v \in V_j \wedge i \preceq j\}$
- full independet set I : $\forall i, I \cap (U_i \cup V_i) \neq \emptyset$
- Projection to downsets
  $D = \{i \in X : I \cap V_i \neq \emptyset\}$. $(2^{2n} - 1)^n$ ways.
- Non-full i.s. $\leq 3^n (2^{2n} - 1)^{n-1}$

Synopsis for #BIS

$$\begin{aligned} \#\text{BIS} &\leq_{AP} \#\text{2-P-WR-C} \\ &\leq_{AP} \#\text{BeachConfigs} \\ &\leq_{AP} \#\text{Downsets} \equiv_{AP} \text{1P1NSAT} \\ &\leq_{AP} \#\text{BIS} \end{aligned}$$

- $\#P_q$-Col $\equiv_{AP}$ #BIS

Intermediate problems

- #BIPARTITEMAXIS: The number of maximum independent sets in a bipartite graph
- #3-P-WR-C: 3 particle WR configurations
- #BIPARTITE q-COL: Number of q-colorings of a bipartite graph

#BIS $\leq_{AP}$ #BipartiteMaxIS

Add to the graph for every vertex v a vertex v' and an edge (v, v'). Now every i.s. in the original graph leads to a maximum i.s. in the new graph.

#BIPARTITEMAXIS $\leq_{AP}$ #3-P-WR-C

- If $B = (X, Y, A)$ is an instance and M is the size of the maximum i.s. construct $G = (V, E)$ with $U_i, 0 \leq i \leq n-1$, $V_i, 0 \leq i \leq n-1$ disjoint sets of size s and k a set of size t, $V = K \bigcup_{i \in [n]} U_i \bigcup_{j \in [n]} V_j$, $E = K^{(2)} \cup \bigcup_{j \in [n]} (V_j \times K) \cup \bigcup \{U_i \times V_j : (x_i, y_j \in A)\}$
- K is a clique. full coloring: K is bichromatic. In a full coloring select y_j iff V_j has the same second color as K, select x_i iff U_i has different colors from K.

#BipartiteMaxIS $\leq_{AP}$ #3-P-WR-C

- $3(2^t - 2)(4^s - 2^s)^k(2^s)^{n-k}(2^s - 1)^l$ combinations for k u_i's and l v_j's.
- Non full colorings and colorings which point to non-maximum independent sets are once again much fewer, and therefore the reduction is complete.