Quantum Complexity

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Structural Complexity $(\mu\Pi\lambda\forall)$

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Overview

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1. Preliminaries
   - Qubits

Overview

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Motivation

- Ordinary computer chips: bits are physically represented by low and high voltages on wires
- There are many other ways a bit could be stored! For example, the state of a hydrogen atom
- The single electron in this atom can either be in the ground state (the lowest energy configuration) or it can be in an excited state (a high energy configuration)
- Ground state: $|0\rangle$. Excited state: $|1\rangle$

Quantum Bit

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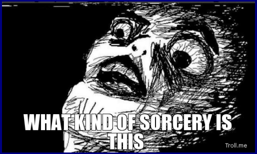


Measurement

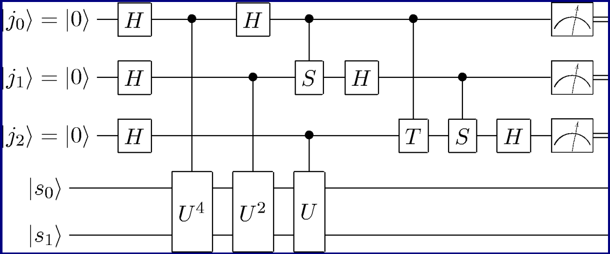
- Measurement of the qubit $|\psi\rangle = a_0|0\rangle + a_1|1\rangle$ gives 0 w.p. $\|a_0\|^2$ and 1 w.p. $\|a_1\|^2$
- Suppose two qubits: $|\phi\rangle = a_0|0\rangle + a_1|1\rangle$ and $|\psi\rangle = b_0|0\rangle + b_1|1\rangle$
- The whole state can be written as $|\phi\psi\rangle = a_0b_0|00\rangle + a_0b_1|01\rangle + a_1b_0|10\rangle + a_1b_1|11\rangle$
- Measurement of two qubits gives 00 w.p. $\|a_0b_0\|^2$, 01 w.p. $\|a_0b_1\|^2$ 10 w.p. $\|a_1b_0\|^2$ and 11 w.p. $\|a_1b_1\|^2$
- A measurement is a normalized projection onto one basis vector of the space and the probability of taking this vector is the square of the norm of this projection

- What if we have two qubits?
- Quantum state: $|\alpha\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$, such that $\Sigma_{x\in\{0,1\}^2}|a_x|^2 = 1$.
- We can see a state of 2 qubits as a unit length column vector in the 4-d complex space
- What if we have 500 qubits?
- The quantum state is a linear superposition of 2^{500} classical states! Way more than the number of elementary particles in the universe!
- Where is all this information stored?
- Can we use this to make faster computers?

Overview

Gates

- We can see Quantum gates as operators applied on one or more qubits
- Those operators are **Unitary**
- U is unitary iff $UU^{\top} = \mathbb{I}$ where $U^{\top}$ is the complex conjugate of U



Gates(2)

- **CNOT** gate:

$$|x, y\rangle \rightarrow |x, x \oplus y\rangle$$

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- **CNOT** gate:

$$|x, y\rangle \to |x, x \oplus y\rangle$$

- **Hadamard** gate:

$$|0\rangle \to \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$|1\rangle \to \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

$$a|0\rangle + b|1\rangle \to \frac{a+b}{\sqrt{2}}|0\rangle + \frac{a-b}{\sqrt{2}}|1\rangle$$

Properties

- Quantum gates unlike Classical gates have the same number of input and output qubits
- Quantum gates do not lose information, which means that Quantum gates...and generaly Quantum Computations are reversible
- A unitary operator preserves the length of a state and the cosine of the angle between 2 states
- So a unitary operator just rotates or mirrors the space of our states

Entanglement

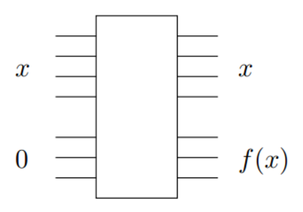
- Suppose we have the state $|\chi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$
- We cannot find states $|\phi\rangle = a_0|0\rangle + a_1|1\rangle$ and $|\psi\rangle = b_0|0\rangle + b_1|1\rangle$ such that $|\phi\rangle|\psi\rangle = |\chi\rangle$
- We say that the qubits in $|\chi\rangle$ are in entanglement
- If we measure only the first qubit we will get 0 w.p. $1/2$ and 1 w.p. $1/2$
- If we measured the first and got b then if we measure the second we will also get b immediately
- No matter the distance between the two qubits!

Parallelism

- A Quantum Computer operates in parallel
- Suppose we have the state $|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle$
- Let's perform the 2-qubit operator CNOT(Controlled-NOT)
- $\mathit{CNOT}|\psi\rangle = \frac{1}{\sqrt{2}}\mathit{CNOT}|00\rangle + \frac{1}{\sqrt{2}}\mathit{CNOT}|10\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$
- We performed the operator on those two states in one step!
- What a pity we don't have access to all that information $\rightarrow$ A measurement will return only 2 bits.

Computing a function

- A quantum circuit that computes a function f is a unitary operator U which takes as input:
  1. n input qubits
  2. The output qubits (in case we have a decision function we have only one output qubit) usually initialized to $|0\rangle$
- And gives as output
  1. The n qubits
  2. The answer in the output qubit
- So $|x\rangle|0\rangle \xrightarrow{U} |x\rangle|0 \oplus f(x)\rangle = |x\rangle|f(x)\rangle$
- **Measurement is always the last step of an algorithm**



- But we are in a quantum world so the input qubits can be in a superposition of many classical inputs
- And of course the output will be a superposition of all the classical outputs

Overview

Quantum Turing Machine

Definition (Quantum Turing Machine - David Deutch, 1985)

A **Quantum Turing machine** (QTM) is a 3-tuple $M = (Q, \Sigma, \delta)$, where Q is a finite set of states, Σ is the alphabet, δ is a state transition "function" and is a mapping from $Qx\Sigma$ to $Qx\Sigma x\{L, R\}xC$, where C is the set of complex numbers.

- $\delta(p, \alpha) = (q, b, d, c)$ represents the following: if M in a state p reads a symbol α (in configuration C_1), then M:
  1. writes symbol b on the square under the tape head
  2. changes the state into q
  3. moves the head on the square in the direction denoted by $d \in \{L, R\}$ (configuration C_2)
- The complex number c is called *amplitude* of this event.
- The probability that M changes its configuration from C_1 to C_2 is $|c|^2$

Overview

The Query Model

- We have an oracle for some $f : \{0,1\}^n \to \{0,1\}$ (decision problems)
- We allow our algorithm to apply arbitrary unitary transformations to its own state, as long as these are defined without reference to the values of f.
- 2 types of queries:
  1. $|x, w\rangle \to |x, w \oplus f(x)\rangle$
  2. $|x\rangle \to (-1)^{f(x)}|x\rangle$
- They can simulate each other with a single query.

Deutsch-Jozsa Algorithm

- We are given a function $f : \{0,1\} \to \{0,1\}$ and wish to compute $f(0) \oplus f(1)$

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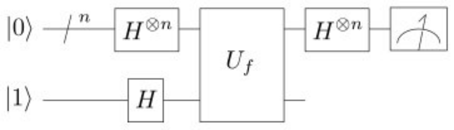
- Factor 2 speedup in computing the XOR of n bits

Deutsch-Jozsa Algorithm(2)

- General version: we are given a function $f : \{0,1\}^n \rightarrow \{0,1\}$, which is either constant or balanced.
- In the classical world we need (worst case) $2^{n-1} + 1$ queries
- In the quantum world, a generalization of the previous algorithm can solve the problem with 1 query!



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Exact Quantum Polynomial Time

Definition (EQP)

EQP is the class of languages $L \subseteq (0,1)^*$, decidable with zero error probability by a uniform family of polynomial-size quantum circuits over some universal family of gates.

- Quantum analogue of P

Bounded Error Quantum Polynomial Time

Definition (BQP)

BQP is the class of languages $L \subseteq (0,1)^*$, decidable with bounded error probability (say $\frac{1}{3}$) by a uniform family of polynomial-size quantum circuits over some universal family of gates.

- Quantum analogue of BPP
- Factoring, DLP $\in$ BQP

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- $\checkmark$So, a Quantum Computer is a least as powerful as a Classical Computer

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- But is that accurate?

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- We can find α_x by looping over all computational paths that contribute amplitude to $|x\rangle$. This requires only polynomial space.
- Then we sum the probabilities of every $|x\rangle$ to take the total accepting probability.

BQP vs PP [Adleman, DeMarrais, Huang - 97]

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- A **PP** problem involves summing up exponentially many terms and then deciding whether the sum is greater or less than some threshold, which is exactly what the Feynman Path Integral does.
- $P_{accept} = \Sigma_{x \in S} |\Sigma_i a_{x,i}|^2$. This is the sum of exponentially many terms, each of which is computable in P! So we can decide in PP whether $P_{accept} \leqslant \frac{1}{3}$ or $P_{accept} \geqslant \frac{2}{3}$

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- BQP is in fact low for PP, meaning that a PP machine achieves no benefit from being able to solve BQP problems instantly.

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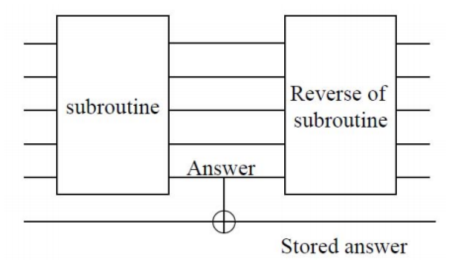
- Obstacle for proving this for BQP: Entanglement (garbage)! The answer of the subroutine depends on its working qubits
- Charles Bennett proposed a smart trick: Uncomputing

Uncomputing



1. Run the subroutine
2. Copy the answer qubit to a separate location
3. Run the subroutine backwards

Overview

Reminder: MA

Definition (MA)

The class of decision problems solvable by a Merlin-Arthur protocol: Merlin (unbounded computational resources) sends Arthur a polynomial-size purported proof that the answer to the problem is "yes". Arthur must verify the proof in BPP so that:

- If the answer is "yes", then there exists a proof such that Arthur accepts w.p. at least 2/3.
- If the answer is "no", then for all proofs Arthur accepts w.p. at most 1/3.

- AM (AM[2]) is the same thing, but this time Arthur goes first and the Merlin answers
- AM[k] = AM[2]

QMA, QCMA

Definition (QMA)

QMA is the class of languages $L \subseteq (0,1)^*$, for which there is a polynomial size quantum circuit A such that $\forall x$

- if $x \in L$ then there is a quantum witness $|w\rangle$ such that $A(x, |w\rangle)$ accepts with probability at least $\frac{2}{3}$
- if $x \notin L$ then for all quantum witnesses $|w\rangle$, $A(x, |w\rangle)$ accepts with probability at most $\frac{1}{3}$

- QMA is the quantum analogue of MA
- **QCMA** stands for: *Quantum Classical Merlin Arthur*.
- In QCMA the witness should be a classical string

Some Bounds

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Quantum Oracle Separation [Aaronson, Kuperberg]

There is a **quantum oracle** A (that is a black box unitary transformation) such that $QCMA^A \neq QMA^A$

Reminder

- **IP**: The class of languages $L \subseteq \{0,1\}$ for which there exists an interaction protocol between *BPP* verifier and an omnipotent prover s.t. $\forall x$:
  1. $x \in L \Rightarrow \exists$ a prover strategy that causes verifier to accept with probability $\geqslant \frac{2}{3}$
  2. $x \notin L \Rightarrow \forall$ prover strategies, verifier accepts with probability $\leqslant \frac{1}{3}$
- **IP = PSPACE** (Shamir)

Quantum Interactive proofs

- The proover and verifier can exchange quantum messages, and are limited by the laws of quantum physics. The number of gates is polynomial.
- **QIP**: The class of languages $L \subseteq \{0,1\}$ for which there exists an interaction protocol between *BQP* verifier (Arthur) and an omnipotent prover (Merlin) s.t. $\forall x$:
  1. If $x \in L$ then the prover can behave in such a way that the verifier accepts with probability at least $\frac{2}{3}$
  2. If $x \notin L$ then however the prover behaves, the verifier rejects with probability at least $\frac{2}{3}$

Quantum Interactive proofs

Theorem (Kitaev,Watrous - 2003)

Any QIP protocol can be made three-round. In other words, all QIP rounds are given by QIP$(1) =$ QMA, QAM $\subseteq$ QIP(2), and QIP$(3) =$ QIP.

Theorem (Jain,Ji,Upadhyay,Watrous - 2009)

QIP = IP = PSPACE

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- It proves that there exists an oracle relative to which BPP $\neq$ BQP

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- Quantumly, Grover's algorithm needs only $2^{\frac{n}{2}}$ on average
- Quantum Computers give quadratic (not exponential) speedup!

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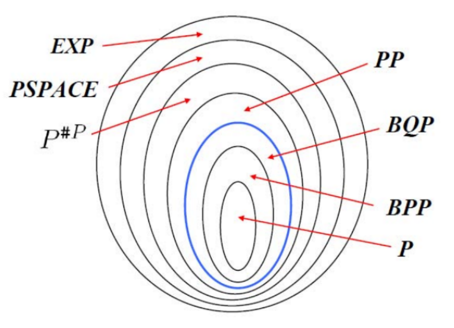
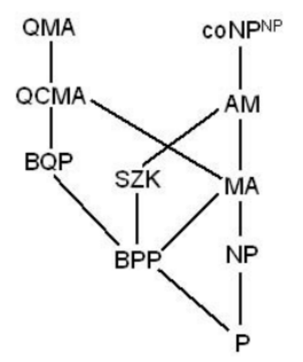
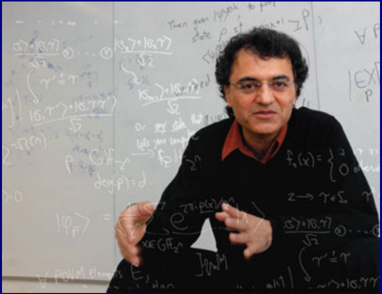
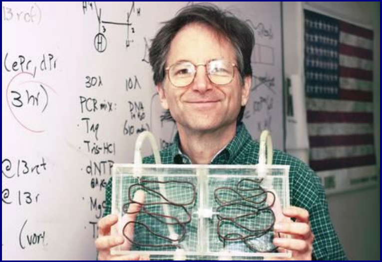
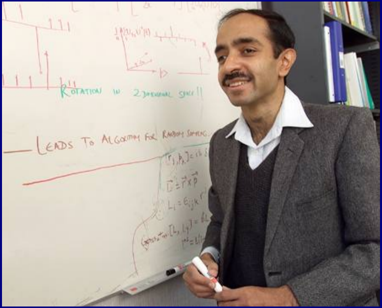
Quantum Complexity Relations

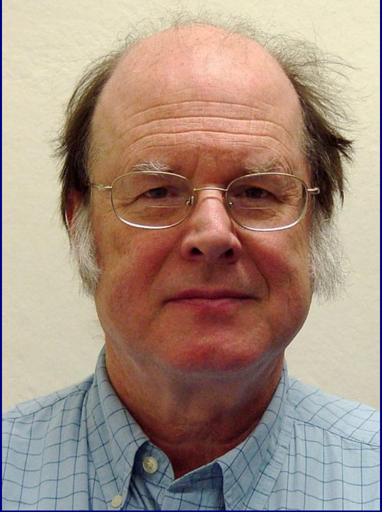







PCR mix:
Tris-HCl


LEADS TO



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