Natural Proofs

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Structural Complexity $(\mu\Pi\lambda\forall)$

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Overview

1 Introduction

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  - algebrization: IP = PSPACE, PCP theorems (Aaronson, Wigderson 2008)

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Theorem

A language L is in P iff L has logspace-uniform polynomial circuits

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Definition (n^c-usefulness)

Let $f : \{0,1\}^n \rightarrow \{0,1\}$, $c \in \mathbb{N}$. Any proof that f does not have n^c-sized circuits can be viewed as defining a predicate $\mathcal{P}$ s.t. $\mathcal{P}(f) = 1$ and $\forall g \in SIZE(n^c)\ \mathcal{P}(g) = 0$

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Constructiveness: we can check in time $2^{\mathcal{O}(n)}$ if $\mathcal{P}(g) = 1$ from g's truth table ✓

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Lemma

If a function f does not have circuits of size $< S$ then at least half of the functions (with the same number of input variables as f) do not have circuits of size $\leqslant S/2 - 3$

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Lemma

If a function f does not have circuits of size < S then at least half of the functions (with the same number of input variables as f) do not have circuits of size $\leqslant$ S/2 -3

Proof.

Let g be a random function then f = $(f \oplus g) \oplus g$. If both g and $f \oplus g$ have circuits of size < S/2 - 3 then f has a circuit of size <S (we need only 5 gates to compute $\oplus$) □

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*subexponentially strong one-way function = one that resists inverting even by a $2^{n^{\epsilon}}$ -time adversary for some fixed $\epsilon > 0$.

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2. For $s \in \{0,1\}^m$ no polynomial time algorithm can distinguish f_s from a random function from $\{0,1\}^m$ to $\{0,1\}$

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- Let $h : \{0,1\}^m \to \{0,1\}$ be an unknown function (it could be either f_s for some s or a random function)
- We will use the natural property $\mathcal{P}$ to tell whether h is a (truly) random function

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- That means that we can distinguish between f_s and a random function with nonnegligible probability in polynomial time.

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For example μ (f) = 1 + the smallest formula size for f

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Theorem

Suppose μ is a formal complexity measure and $\mu(f) \geqslant S$ for some f and some large number S. Then $Pr[\mu(g) \geqslant S/4] \geqslant 1/4$.

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Suppose μ is a formal complexity measure and $\mu(f) \geqslant S$ for some f and some large number S. Then $Pr[\mu(g) \geqslant S/4] \geqslant 1/4$.

Proof.

For random g let $h = f \oplus g$ so $f = h \oplus g = (h \wedge \bar{g}) \vee (g \wedge \bar{h})$. If $Pr[\mu(g) < S/4] > 3/4$ then $\mu(h), \mu(g), \mu(\overline{g}), \mu(\bar{h}) < S/4$, so $\mu(f) < S$, but that is absurd. □

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That means that if one way functions exist we cannot prove P $\neq NP$ that way

Overview

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- But diagonalization is a relativizing proof technique...

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Definition (promiseMA)

Let f be a promise problem. f $\in$ promiseMA if for every $x \in \{0,1\}^*$ $\exists$ polynomials p,q and a polynomial time algorithm A s.t.:

$f(x) = 1 \Rightarrow \exists y \in \{0,1\}^{q(|x|)}, \exists z \in \{0,1\}^{p(|x|)} Pr[A(x,y,z) = 1] \geqslant 2/3$

$f(x) = 0 \Rightarrow \exists y \in \{0,1\}^{q(|x|)}, \exists z \in \{0,1\}^{p(|x|)} Pr[A(x,y,z) = 1] \leqslant 1/3$

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the proof of the first theorem uses diagonalization and the proof of the second theorem uses the first result

Moral

- We showed that we can use a natural property (one that holds for a nonnegligible fraction of boolean functions and can easily be checked) to distinguish a pseudorandom function from a truly random function

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- It is interesting that we used computational complexity to shed light on a metamathematical question about computational complexity
- But remember that we used a condition (existence of one-way functions) that is stronger than $P \neq NP$...

Bios

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Alexander Razborov (1963-) 
Nevanlinna Prize (Approximation method, 1990), Gödel Prize (Natural Proofs, 2007)

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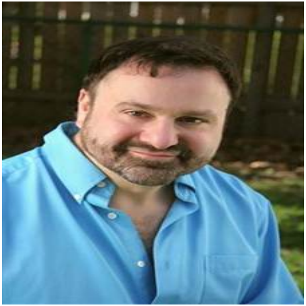

Carnegie Mellon University

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THANK YOU!