Measure & Dimension in Complexity Classes

Konstantinos X. Koiliaris

$\mu \prod \lambda\forall$

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...But let us start from the beginning; i.e., the motivation!

Introduction: General Motivation

Lutz was primarily interested in:

- the NP problem (What is the power of nondeterminism?)
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In trying to investigate these problems, he developed the three analytic tools mentioned (Resource-bounded Category, Measure and Dimension).

These are complexity-theoretic generalizations of classical Baire category, **Lebesgue measure**, and **Hausdorff dimension** (fractal dimension), respectively. Each provides a mathematical means of quantifying the sizes of subsets of various complexity classes.

Introduction: Important Figures

Before we begin formally, I would like to present the most important figures of this area:

- Jack Lutz (1987 @ Caltech: Alexander S. Kechris)
- Claus-Peter Schnorr (1967 @ Universitt des Saarlandes: Günter Hotz)
- Charles H. Bennett (1970 @ Harvard: David Turnbull and Berni Alder)
- John Gill (1972 @ UC Berkeley: Manuel Blum and Robert W. Robinson)
- Klaus Wagner (D) (1976 @ Friedrich-Schiller-Universitt Jena: Karl Dörge)
- Heribert Vollmer (1994 @ Julius-Maximilians-Universität Wrzburg: Klaus Wagner)
- Ronald Vernon Book (D) (1969 @ Harvard: Sheila Adele Greibach)
- Juris Hartmanis (1955 @ Caltech: Robert P. Dilworth)

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Introduction: Motivation (1)

Separations of complexity classes $\mathcal{C} \subsetneq \mathcal{D}$, although they may be very hard to obtain, are in some sense not very pleasing. In principle, they only establish the existence of a single language L in $\mathcal{D}$ which is not a member of $\mathcal{C}$. A more convincing argument would be to show that $\mathcal{D}$ is much *larger* than $\mathcal{C}$, i.e., that languages like L are **frequent**.

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In the famous case of P versus NP, the above objection does not really hold. Separating P from NP would prove that none of the hundreds of NP-complete languages can be solved in polynomial time. But how can we formalize a statement like "most languages in NP are NP-complete"? More generally, we would like to have a **measure** for the frequency with which a given property P occurs within a complexity class.

Introduction: Motivation (2)

Towards this direction, let us define
$\Delta = \{x_0, x_1, x_2, ... \mid (\forall i)[x_i = 0 \vee x_i = 1]\} = \{0,1\}^\infty$ the set of all infinite binary sequences.

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Intuitively, one can think of the Lebesgue measure of a set $\mathcal{A} \subseteq \{0,1\}^\infty$ as the probability that we end up with a sequence in $\mathcal{A}$ when we flip an unbiased coin to determine each of the bits of the sequence independently.

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Intuitively, one can think of the Lebesgue measure of a set $\mathcal{A} \subseteq \{0,1\}^\infty$ as the probability that we end up with a sequence in $\mathcal{A}$ when we flip an unbiased coin to determine each of the bits of the sequence independently. We would say that a property P is frequent if the Lebesgue measure of the class of characteristic sequences of languages that satisfy P is large.

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Since, our classes are all **countable**, therefore the Lebesgue measure would *always* be 0.

Introduction: Motivation (4)

The remedy to this situation was to consider a restriction of Lebesgue measure in the mathematical sense and at the same time, keep the nice properties of the Lebesgue measure: subset of a small set is small and that the union of two small sets is small.

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The remedy to this situation was to consider a restriction of Lebesgue measure in the mathematical sense and at the same time, keep the nice properties of the Lebesgue measure: subset of a small set is small and that the union of two small sets is small.

This approach is based on a characterization of Lebesgue zero measure using **martingales**.

Definition: Martingales (1)

A *martingale* is abstractly defined as a function $d : \{0,1\}^* \to [0,\infty)$ satisfying the following law:

$$d(w) = \frac{d(w0) + d(w1)}{2}, \text{ for every } w \in \{0,1\}^*,$$

where $w0$ is the concatenation of w and 0.

A martingale *succeeds* on a sequence $\omega \in \{0,1\}^\infty$ if:

$$d(\omega) = \limsup_{w \sqsubseteq \omega, w \to \omega} d(w) = \infty$$

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Martingales yield the following characterization.

Definition: Resource-bounded Measure (1)

Theorem

A class $\mathcal{C}$ has Lebesgue measure zero iff it can be covered by a martingale.

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Lutz obtained a resource-bounded measure variant by putting resource bounds on the martingales.

He showed that for certain classes of time bounds one loses no generality by requiring that martingales themselves have values such that all digits are output within the time bound. That means that, given any martingale d meeting the original definition of computability within the time bound, one can obtain a rational-valued d' computable within that bound such that $S^{\infty}[d] \subseteq S^{\infty}[d']$.

Definition

Let Δ be a complexity class. A class $\mathcal{C}$ of languages has Δ-measure zero, written $\mu_{\Delta}(\mathcal{C}) = 0$, if there is a martingale d computable in Δ such that $\mathcal{C} \subseteq S^{\infty}[d]$.

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Because complexity bounds on languages we want to analyze will naturally be stated in terms of n, we prefer to use n for the martingale complexity bounds. The following correspondence is helpful:

$$\text{Lutz's "}p\text{"} \sim N^{O(1)} = 2^{O(n)} \sim \text{measure on } \mathrm{E}$$
$$\text{Lutz's "}p_2\text{"} \sim 2^{\log N^{O(1)}} = 2^{n^{O(n)}} \sim \text{measure on } \mathrm{EXP}$$

Definition: Resource-bounded Measure (3)

Since we measure the time to compute $d(w)$ in terms of n, we write "μ_{E}" for E-measure and "μ_{EXP}" for EXP-measure, and generally μ_Δ for any Δ. Similarly, we define:

Definition

A class $\mathcal{C}$ has measure Δ-measure one ($\mu_\Delta(\mathcal{C}) = 1$), if $\mu_\Delta(\Delta \backslash \mathcal{C}) = 0$

Applications

Theorem (Lutz 1990)

$\mathrm{P}/Poly$ $[: \mathrm{PSIZE}] \bigcap \mathrm{ESPACE}$ *is a measure 0 subset of* ESPACE.

This means that the phenomenon of not having polynomial-size circuits is **very** typical of problems in ESPACE, in the sense of measure.

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This means that the phenomenon of not having polynomial-size circuits is **very** typical of problems in ESPACE, in the sense of measure.

Theorem (Lutz 1992)

Almost every problem in ESPACE *has essentially maximum circuit-size complexity almost everywhere.*

This proves that the *Shannon effect* holds with **full force** in ESPACE.

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In spite of the power of *resource-bounded measure*, there are certain inherent limitations to the amount of quantitative information that it can provide in computational complexity. One of these limitations arises from the resource-bounded Kolmogorov zero-one law:

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Another limitation arises from the simple fact that even a measure 0 subset of a complexity class may have internal structure that we would like to elaborate on quantitatively.

Both of these limitations were already present in the classical Lebesgue measure theory.

Dimension comes into play

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In this presentation, we are interested in the case where the metric space is the Cantor space **C**, consisting of all decision problems (i.e., all languages $A \subseteq \{0,1\}^*$). In this case the Hausdorff dimension of a set $X \subseteq$ **C** is a real number $\dim_H(X) \in [0,1]$.

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The Hausdorff dimension is monotone, with $\dim_H(\emptyset) = 0$ and $\dim_H(\mathbf{C}) = 1$. Moreover, if $\dim_H(X) < \dim_H(\mathbf{C})$, then X is a measure 0 subset of **C** . Hausdorff dimension thus overcomes both of the limitations mentioned in the previous slide.

Resource-bounded Dimension (1)

Based on the classical definition just given, Lutz defines *resource-bounded dimension*, which is a complexity-theoretic generalization of classical Hausdorff dimension.

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This generalization takes place in two steps:

- Initially, he proves a new characterization of classical Hausdorff dimension in terms of *gales*, which are a natural generalization of the martingales.
- Consequently, he generalizes classical dimension by introducing a resource bound Δ and requires the gales to be Δ-computable.

Resource-bounded Dimension (2)

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You can read everything in details at "*Dimension in Complexity Classes*" by J. Lutz 2000.

We will finish this chapter with a few understandable applications.

Applications (1)

We will show that for each $0 \leq s \leq 1$ there is a natural set X that has dimension equal to s in each of the exponential classes E and EXP.

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That is, $\mathsf{freq}_A(n)$ is the fraction of the first n strings in $\{0,1\}^*$ that are elements of A. For $a \in [0,1]$, we define the sets:

$$\mathsf{FREQ}(a) = \{A \in \mathbf{C} \mid \lim_{n\to\infty} \mathsf{freq}_A(n) = a\}$$
$$\mathsf{FREQ}(\le a) = \{A \in \mathbf{C} \mid \liminf_{n\to\infty} \mathsf{freq}_A(n) = a\}$$

Applications (2)

The set FREQ$(\leq a)$ is precisely the set X promised in the previous slide.

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The results also use the binary entropy function defined by Shannon:

$$\mathcal{H} : [0,1] \to [0,1] : \mathcal{H}(a) = a \log \frac{1}{2} + (1-a) \log \frac{1}{1-a},$$

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where $\mathcal{H}(0) = \mathcal{H}(1) = 0$.

In a long proof, Lutz showed that for every real number $a \in [0, \frac{1}{2}]$,

$$\dim\left(\mathrm{FREQ}(\leq a) \mid \mathrm{E}\right) = \mathcal{H}(a)$$

and

$$\dim\left(\mathrm{FREQ}(\leq a) \mid \mathrm{EXP}\right) = \mathcal{H}(a),$$

Applications (3)

Theorem (Lutz 1990)

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This means that the phenomenon of not having polynomial-size circuits is **very** typical of problems in ESPACE, in the sense of dimension.

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These applications are interesting because they show that resource-bounded dimension interacts informatively with information theory and Boolean circuit-size complexity. However, they are clearly only the beginning.

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Classical Hausdorff dimension is a sophisticated mathematical theory that has emerged as one of the most important tools for the investigation of fractal sets. Many sets of interest in computational complexity seem to have "fractal-like" structures. Resource-bounded dimension will be a useful tool for the study of such sets.

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Thank you!