The Butterfly, Cube-Connected-Cycles and Benes Networks

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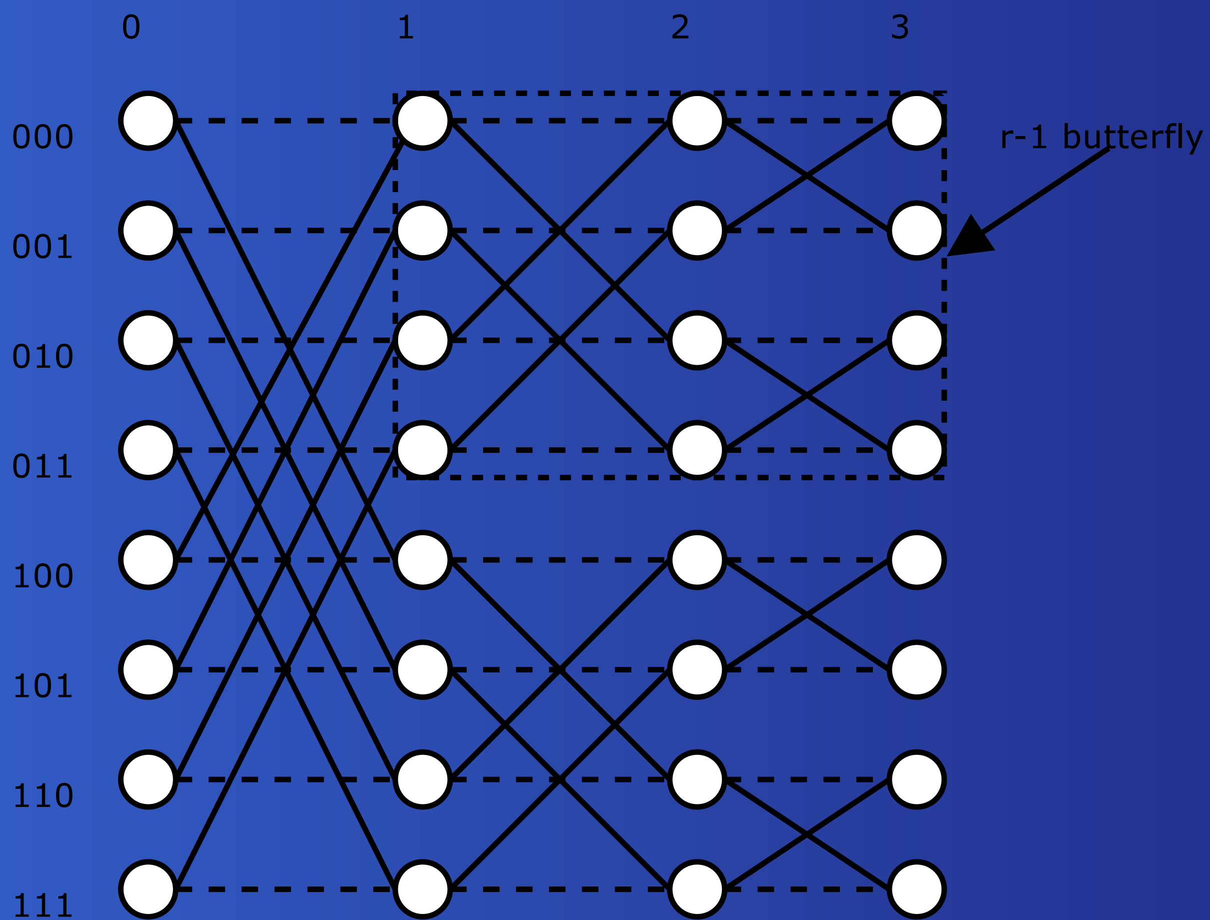
Introduction

- Hypercubes are computationally powerful. Their drawback is that node degree increases with the size of the network.
- Butterfly, CCC and Benes networks are variations of the hypercube with constant degree.
- All three are computationally equivalent, and *universal*.
- They can simulate simple hypercube algorithms with a constant degree slowdown.

The butterfly (i)

- An r-dimensional butterfly has $(r+1)2^r$ nodes corresponding to pairs (w,i) where w is an r-bit binary number and i is the level $0 \leq i \leq r$.
- Two nodes (w,i), (w',i') are linked iff $i' = i+1$ and $w = w'$ or w and w' differ only in the i'th bit. In total $r2^{r+1}$ edges.
- Butterflies with $N(\log N + 1)$ nodes can be viewed as hypercubes with N nodes (if we collapse the rows).

Example of a 3-level butterfly



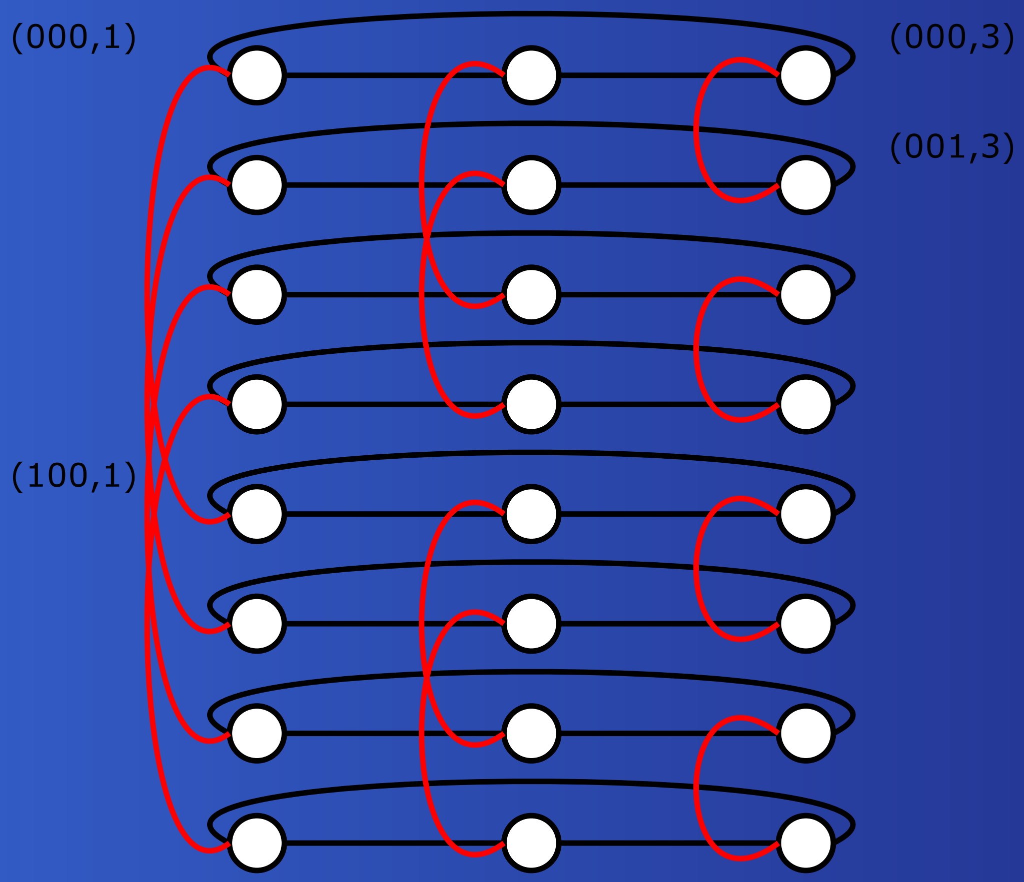
The butterfly (ii)

- Advantages: Recursive structure, unique paths from level 0 to level r ($\longrightarrow$ diameter $O(\log N)$), bisection width $\Theta(N/\log N)$.
- Variation: wrapped butterfly. Level 0 and r are merged. Computationally equivalent to simple butterfly. Useful property: symmetry under cyclic shifts of the levels.

The Cube-Connected-Cycles

- By replacing every node of an r-dimensional hypercube with a cycle of r nodes we get a CCC. Nodes are labelled (w, i), where w is the hypercube node label and i is the cycle label.
- CCCs with $r2^r$ nodes can simulate hypercubes with 2^r nodes with $O(r)$ slowdown.
- CCCs $\equiv$ wrapped butterflies: embeddable with dilation 2. Diameter and bisection width are the same.

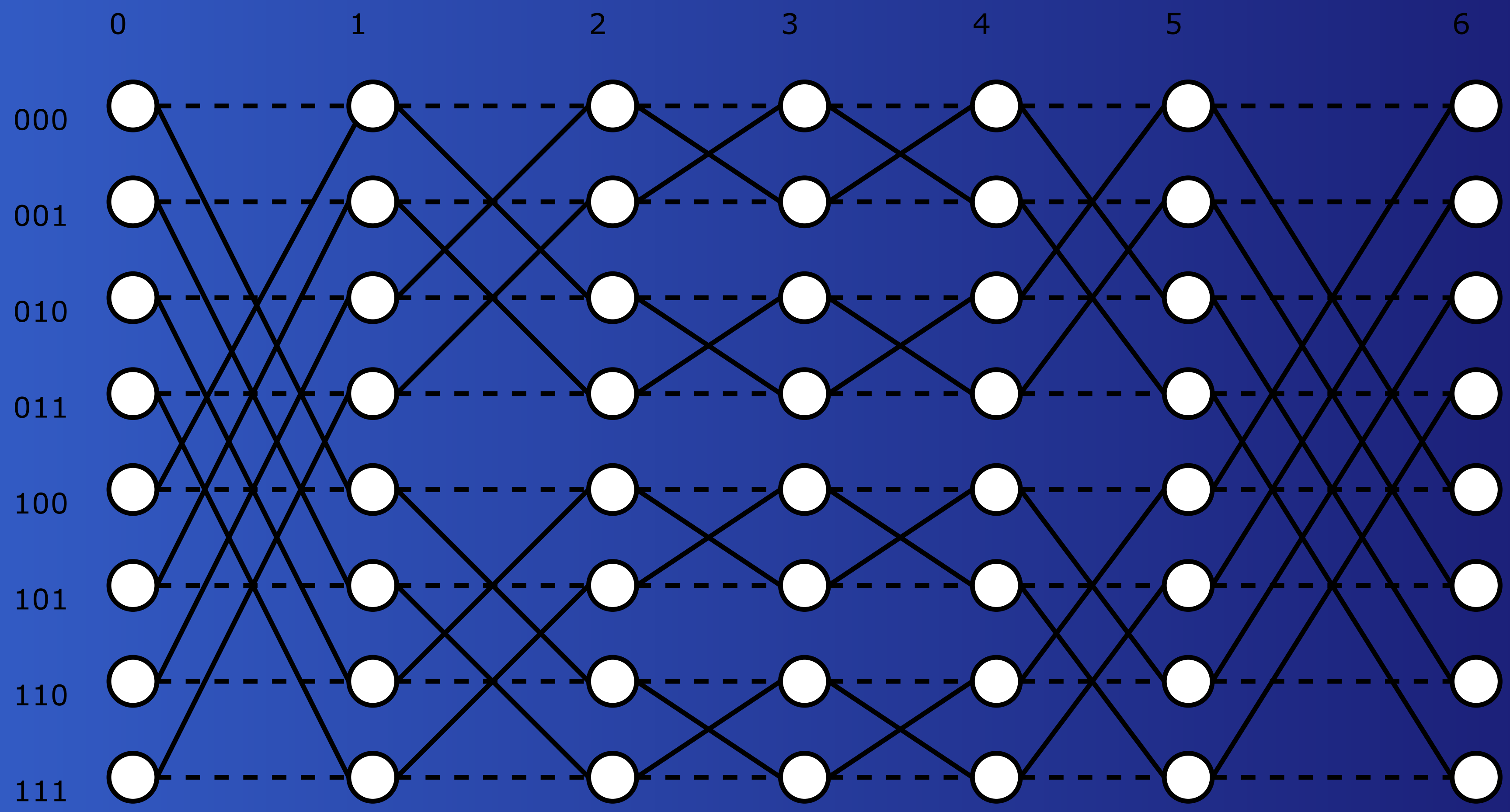
CCCs $\equiv$ wrapped butterfly



The Benes Network

- Two back to back butterflies. $(2r+1)2^r$ nodes in total.
- Interesting property: Rearrangeable network. Useful in the simulation of arbitrary networks.
- Rearrangeability: We can connect all level-0 nodes one to one to all level-$2r$ nodes using node disjoint paths (true for all permutations).

Example of a Benes network



Simulation of Arbitrary networks

- An N-node (wrapped) butterfly can simulate any bounded degree N-node network with at most $O(\log N)$ slowdown.
- This result also applies to simple butterflies, CCCs and Benes networks, since these can all simulate each other with a constant factor loss of efficiency.
- Hence, butterflies are *universal*

Simulation - Proof sketch (i)

- Lemma: Given an N-node wrapped butterfly and a permutation $\pi : [0 \ldots N] \rightarrow [0 \ldots N]$ there is a way of moving at most one packet from every node i to node $\pi(i)$ within $3 \log N$ steps without causing congestion in edges or nodes.
- Solution: Routing in 3 phases (similar to $r \times 2^r$ array). Phases 1 and 3 permute packets in rows, using row edges. Phase 2 permutes packets in columns in at most $2r$ steps (rearrangeability of Benes).

Simulation - Proof sketch (ii)

- To simulate a step of a network of maximum degree d we solve d packet routing problems using the previous lemma, after mapping each node of the network to a node of the butterfly.
- Problem: More than one packet may originate at or be headed to the same node in the same step (at most d).
- Solution: Use bipartite edge coloring to partition communication requests in d groups.

Normal Hypercube Algorithms (i)

- Using previous simulation $O(\log^2 N)$ slowdown.
- Normal Hypercube algorithms can be simulated with a constant slowdown.
- Normal: Using only one dimension at each step and using consecutive dimensions in consecutive steps.
- All the mesh-of-trees algorithms described in 3.1 are normal.

Normal Hypercube Algorithms (ii)

- N-node hypercube with $N = r2^r$ is embedded to N-node CCC by mapping node $v = v_1 v_2 \ldots v_{r+\log r}$ to node $f(v)$
- If k is the first dimension used by the algorithm then set $s = v_{k+\frac{r}{2}+1} \ldots v_{k+\frac{r}{2}+\log r}$
- Remove the bits of s from v to form $u = v_k v_{k+1} \ldots v_{k+\frac{r}{2}} v_{k+\frac{r}{2}+\log r+1} \ldots v_{k-1}$ (a cyclic shift of v)
- Set $f(v) = (\lambda_s(u), s+1)$ where $\lambda_s()$ means s cyclic shifts to the right

Normal Hypercube Algorithms (iii)

- In the next step dimension $k-1$ or $k+1$ is used.
- If $k+1$ is used, shift processes from nodes (w,i) to nodes $(w,i+1)$
- Fine until we use a dimension outside of $[k-\frac{r}{2}+1, k+\frac{r}{2}]$ (different s). Produce a new mapping - only happens once in every $\frac{r}{2}$ steps and costs at most $O(r)$.
- Total time (if T is hypercube time): $2T+\lfloor\frac{2T}{r}O(r)\rfloor=O(T)$.

Containment and Simulation Results

- r-dimensional wrapped butterflies and r-dimensional CCCs are Hamiltonian for $r \geq 2$
- Therefore N-node linear arrays can be simulated by N-node wrapped butterflies or CCCs without slowdown.
- Two-dimensional arrays can only be embedded with dilation $\Omega(\log N)$ (the worst possible)
- However, $O(1)$-dimensional arrays can be simulated with constant slowdown.