Reductions & *NP*-completeness

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Outline

0/1 Integer Programming

3-colorability

Hamilton Path (HP)

Traveling Salesman Problem (TSP)

Reducing 3-SAT to 0/1 IP

Definition 1.1 (0/1 IP).

Input: an integer matrix C and vector b.
Output: decide if there is a 0/1 vector x such that: $Cx \geq b$.

- 0/1 IP $\in$ *NP*(why?)
- We choose 3-SAT as our known *NP*-complete problem and consider the formula:

$$\phi = C_1 \wedge C_2 \wedge \ldots \wedge C_m$$

with literals $x_1, \ldots x_n$

We will construct our $m \times n$ matrix C : $c_{ij} = \begin{cases} 1, & \text{if } x_j \in C_i \\ -1, & \text{if } \bar{x_j} \in C_i \\ 0, & \text{otherwise} \end{cases}$ and

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3-SAT $\leq^{P}$ 0/1 IP

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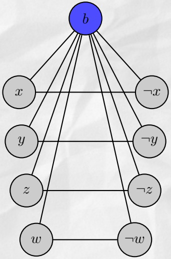

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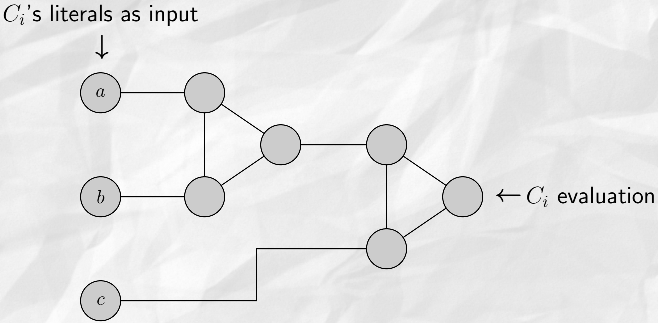
The gadget: a color-driven "or" gate



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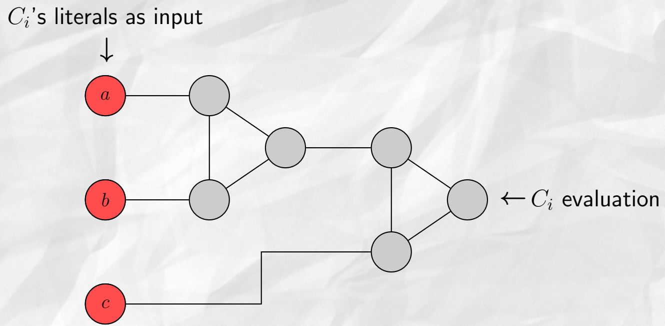

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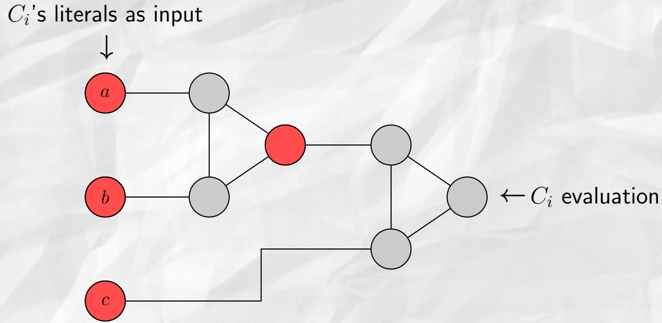

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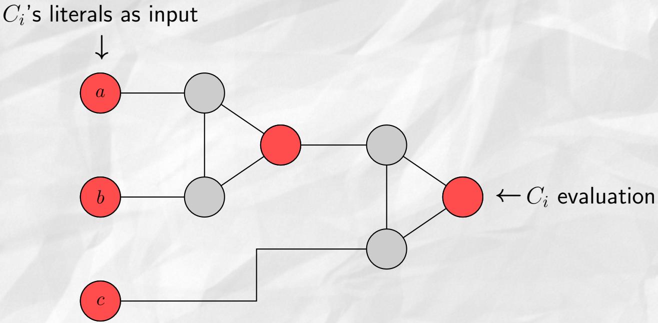

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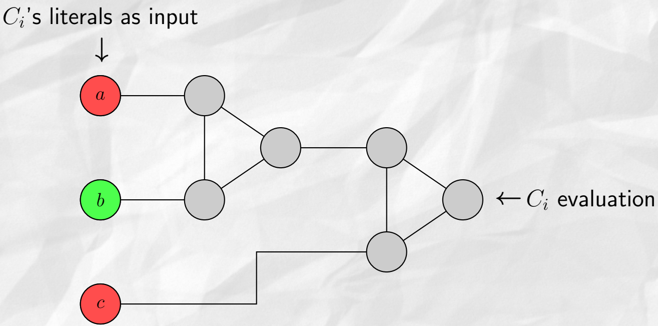

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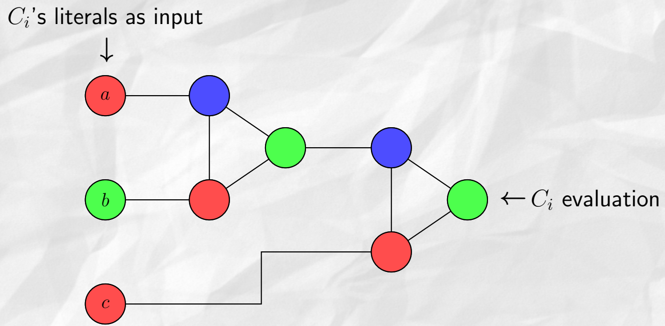

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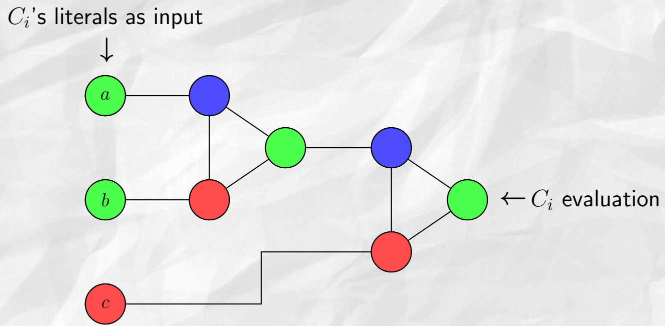

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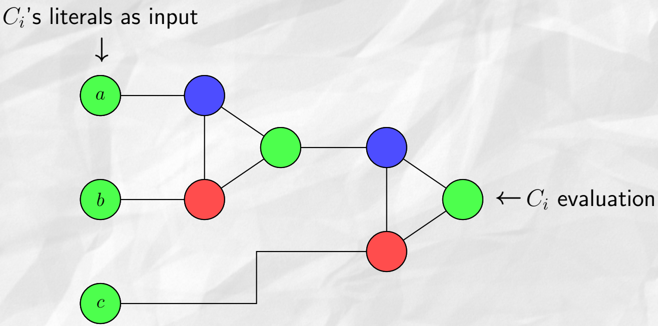

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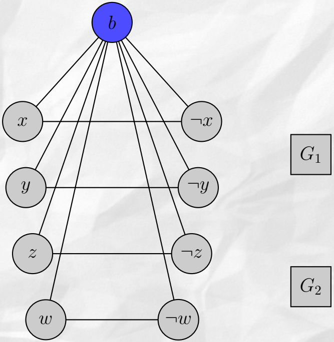

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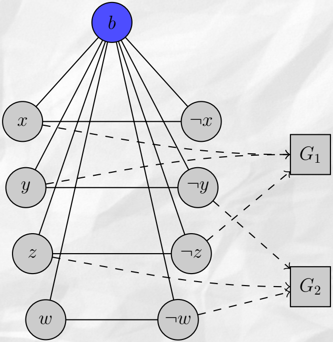

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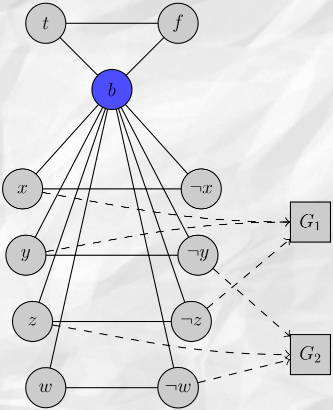

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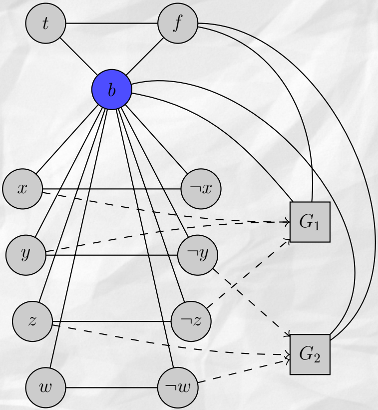

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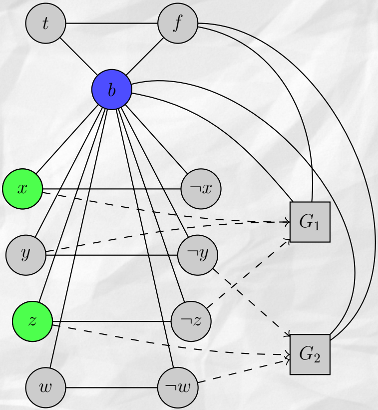

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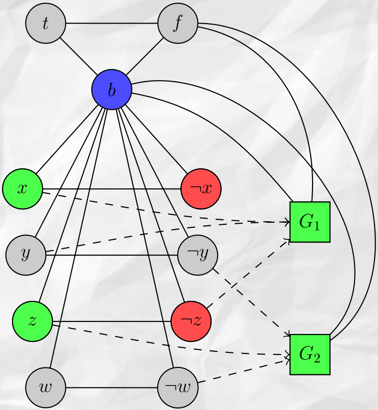

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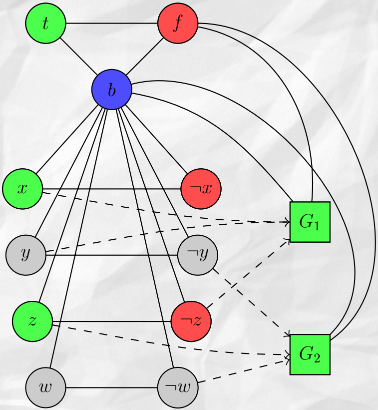

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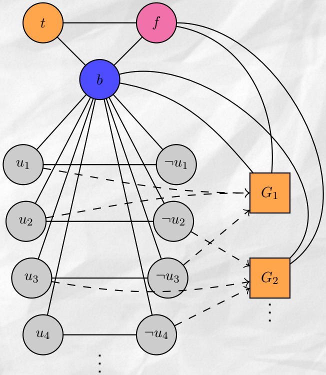
Checking the "if and only if"

- Now let G_ϕ be 3-colorable.



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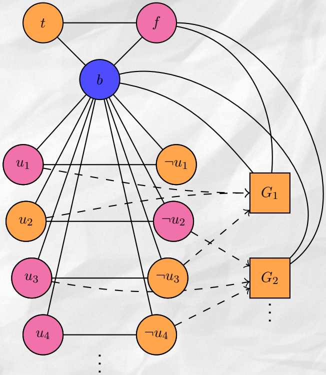

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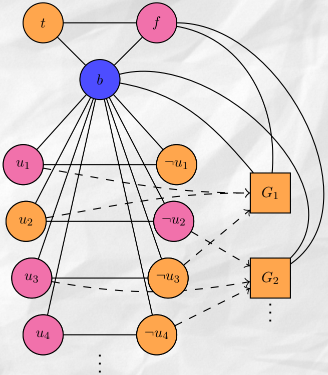

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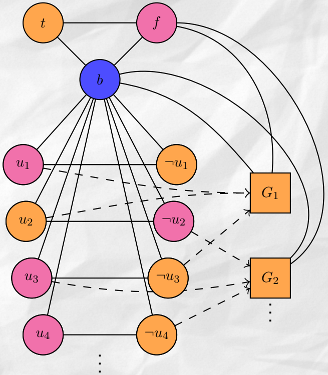

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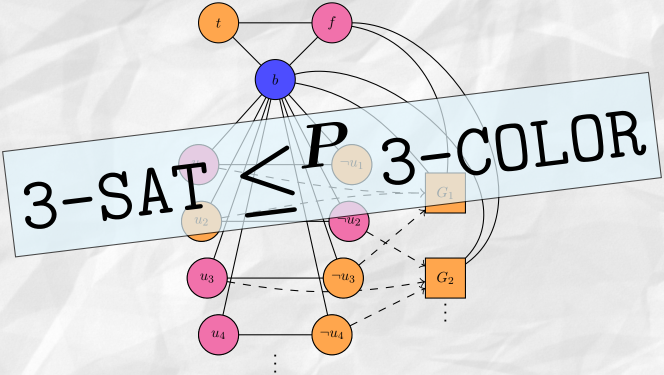

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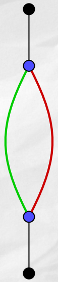

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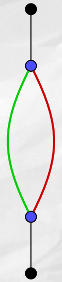

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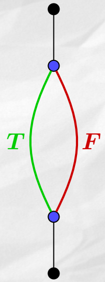

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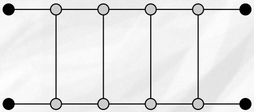
- They sure translate to an evaluation "True" of "False" for the literal.

Gadgets (2/3)

The consistency gadget - an "xor" gate



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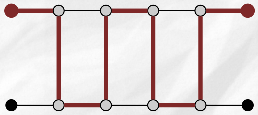

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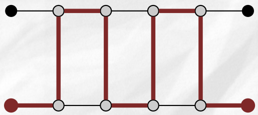

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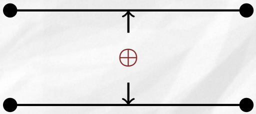

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- That “exclusive or” functionality will be the hint for **gadget 3** to prove useful.

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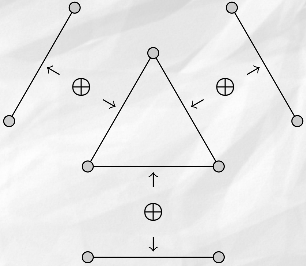

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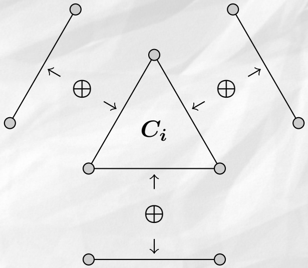

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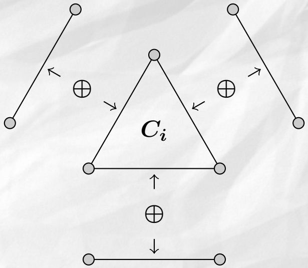

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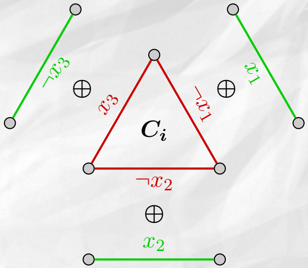

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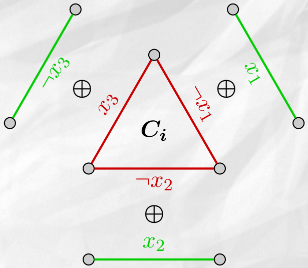

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- **Then the clause is True, or else there would be no Hamilton Path!**

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Let $\phi = (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3)$

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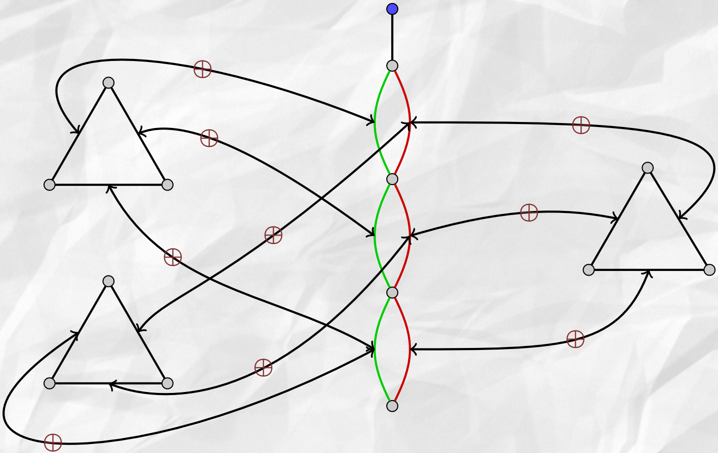

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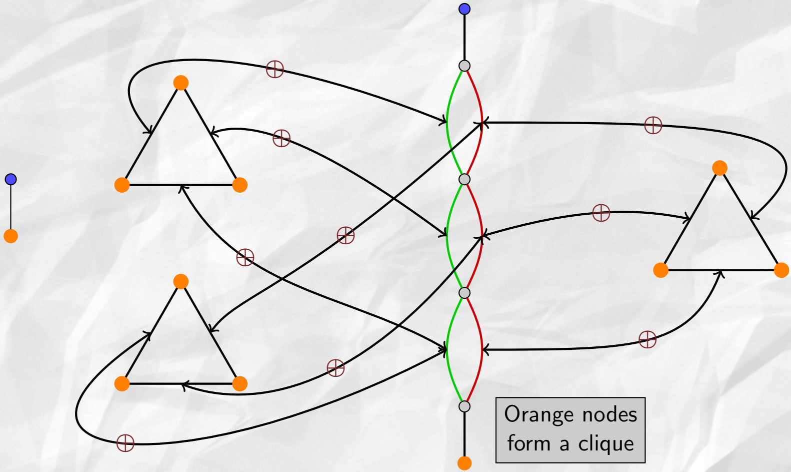

Orange nodes form a clique

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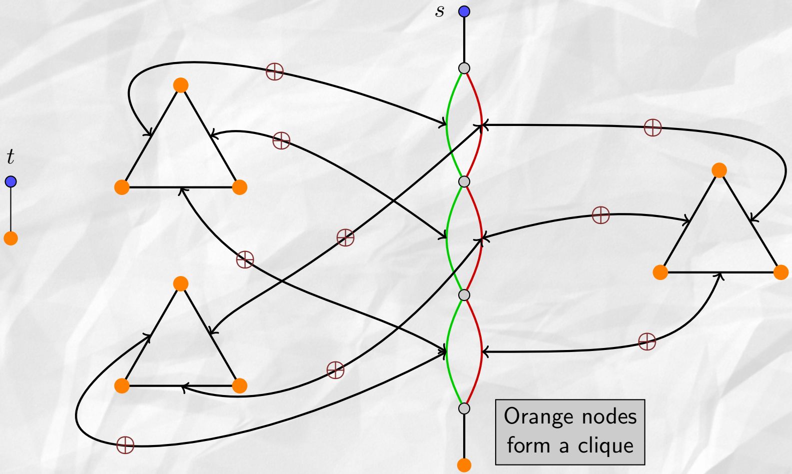

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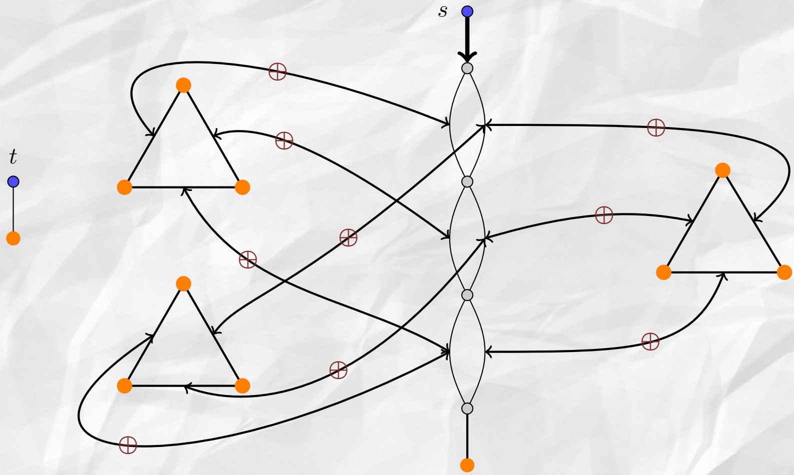

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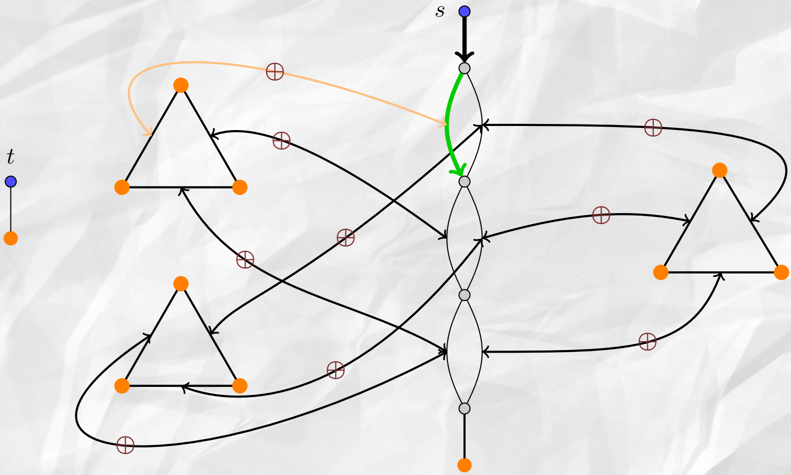

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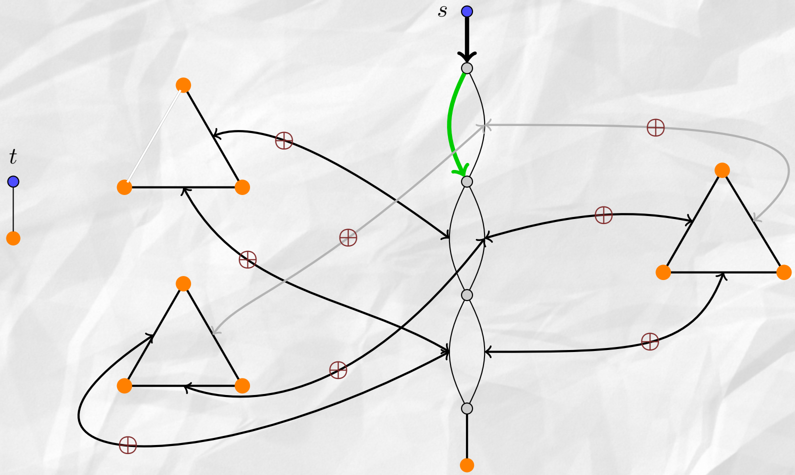

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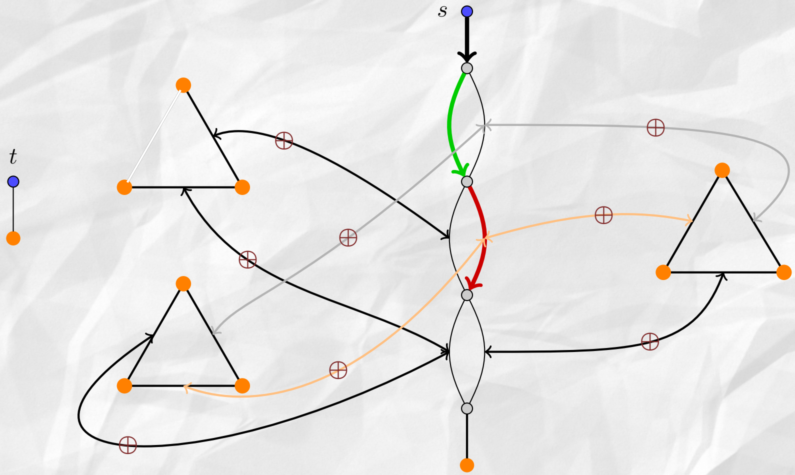

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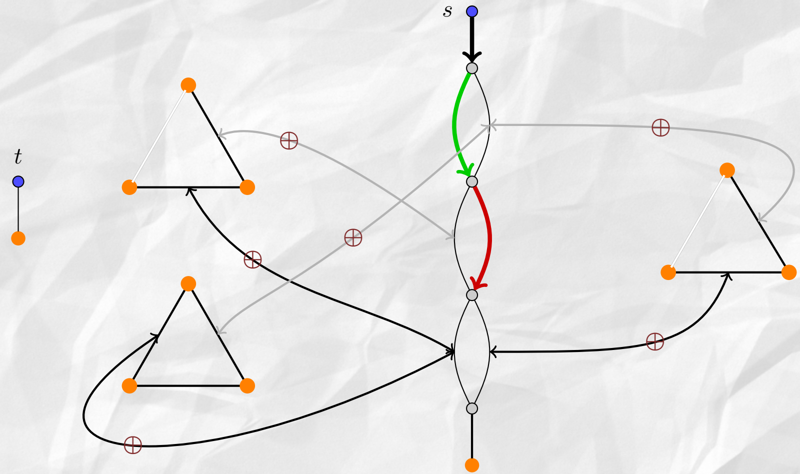

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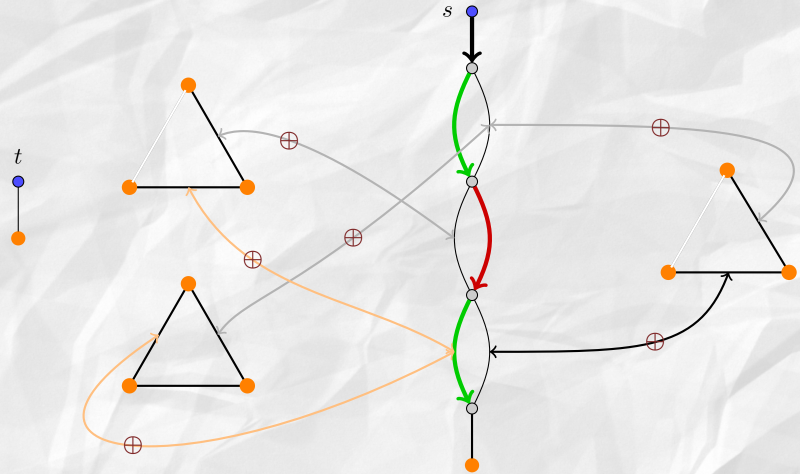

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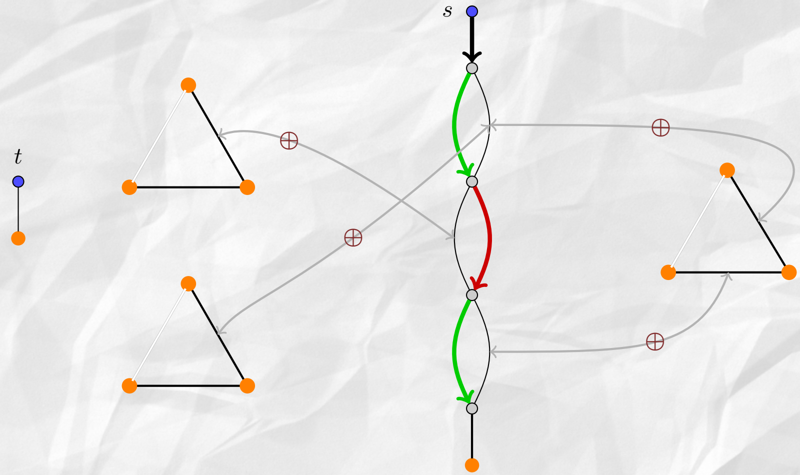

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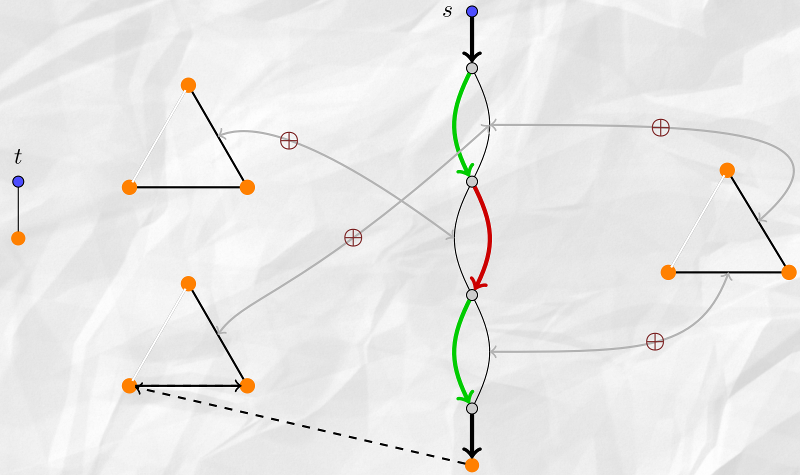

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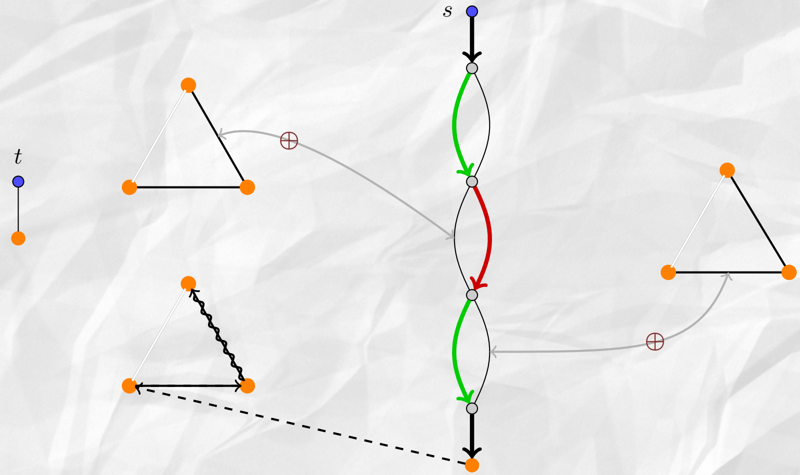

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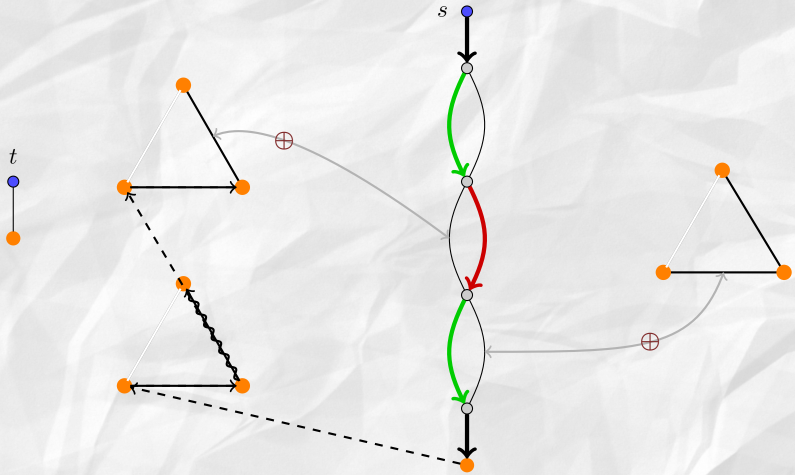

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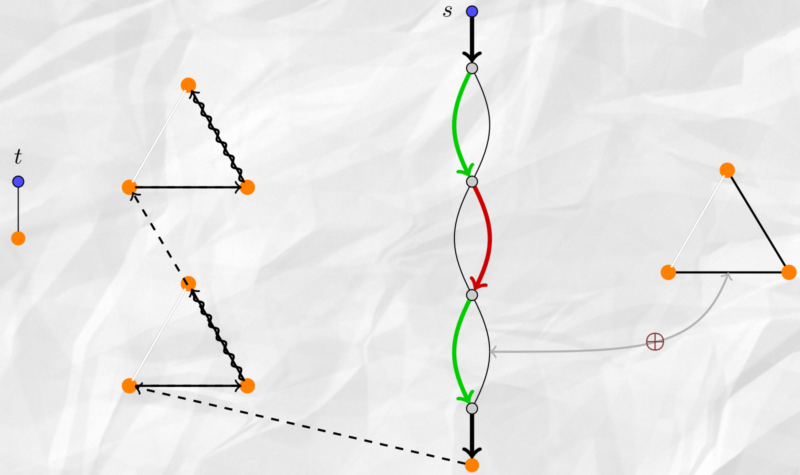

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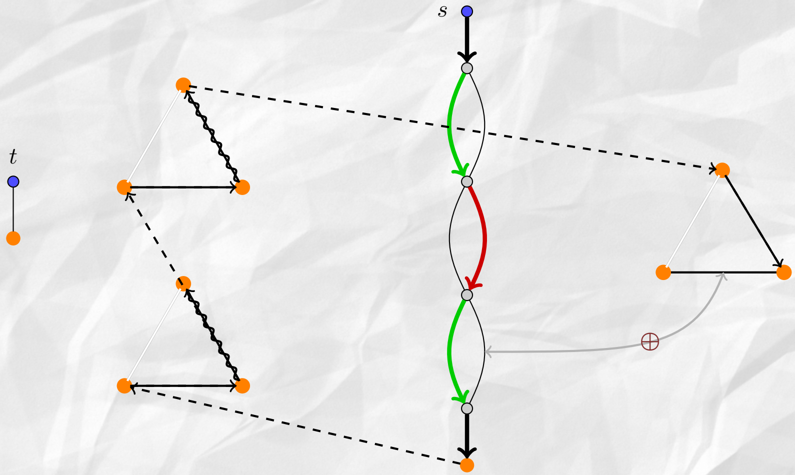

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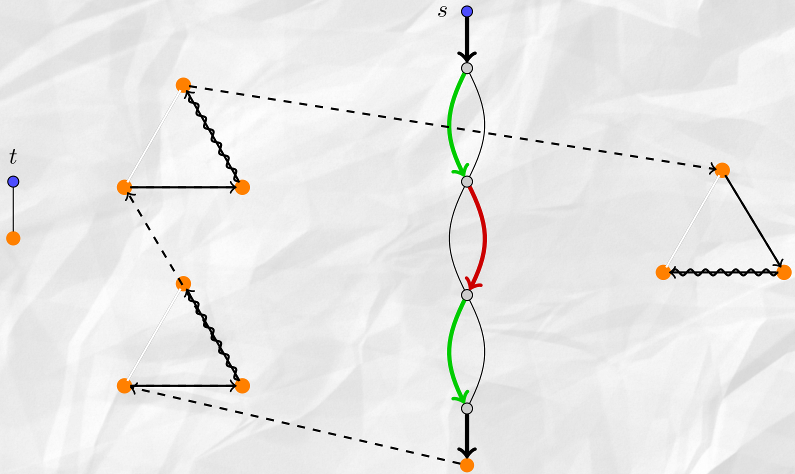

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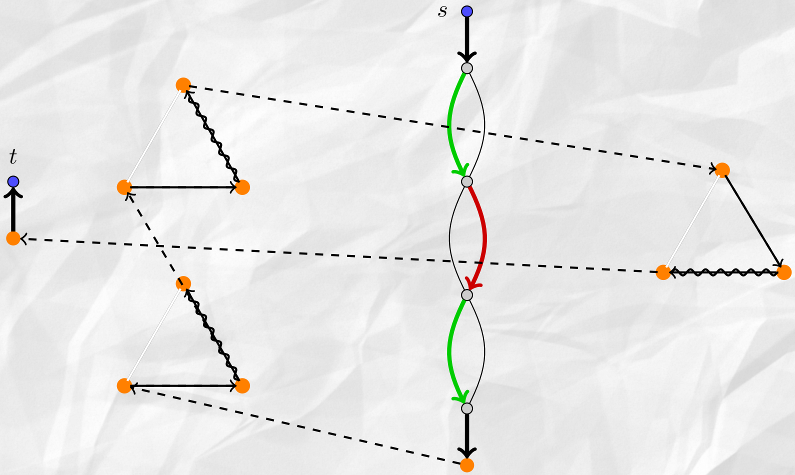

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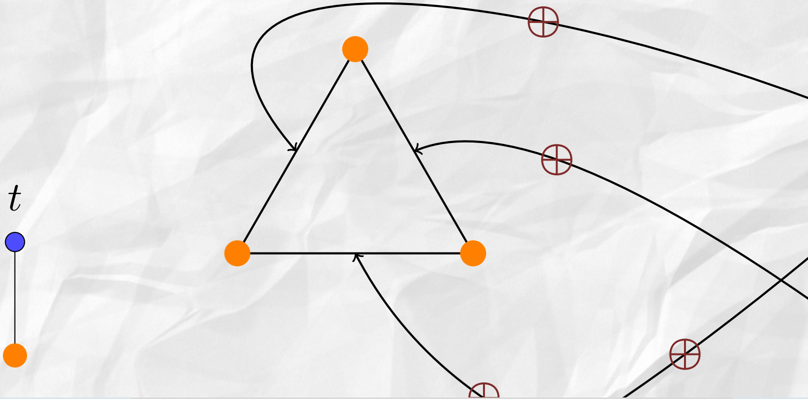
Remember the costraint gadget.



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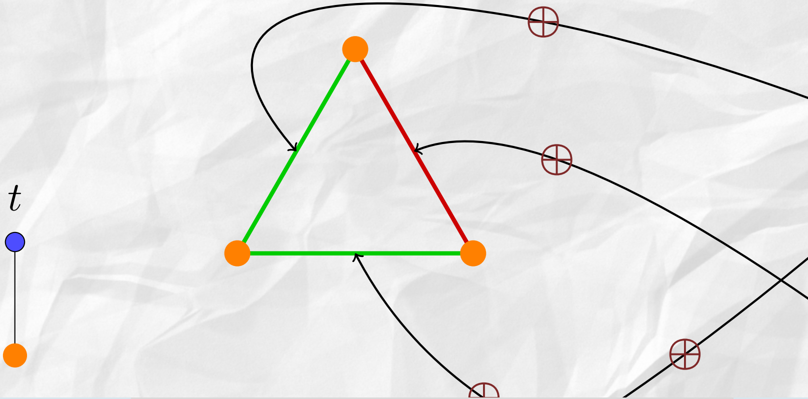

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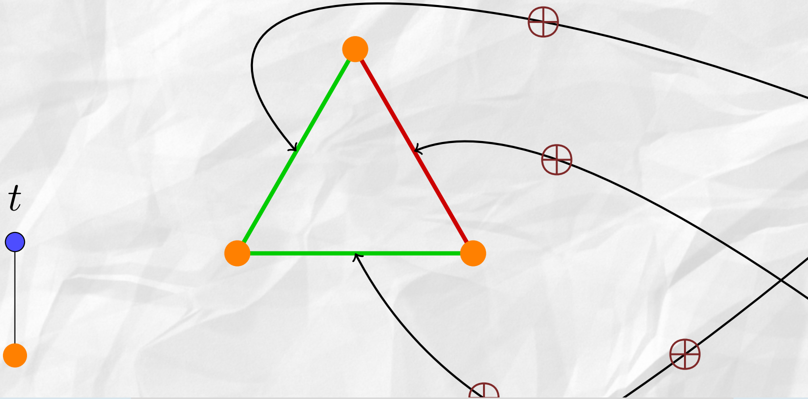

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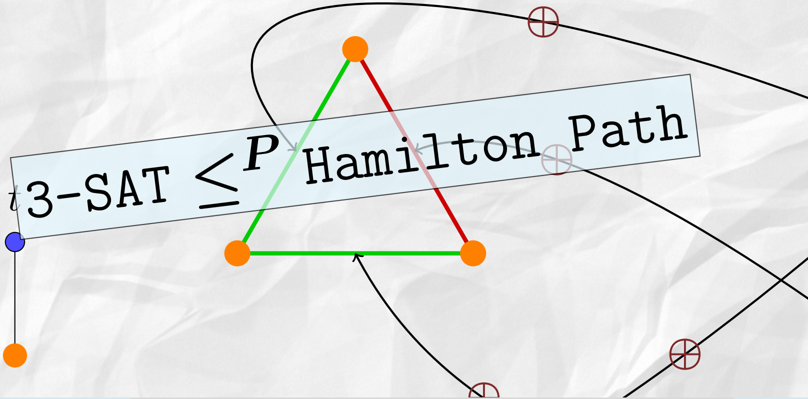

Outline

0/1 Integer Programming

3-colorability

Hamilton Path (HP)

Traveling Salesman Problem (TSP)

The Traveling Salesman Problem

Definition 4.1 (TSP).

Given a set of n cities and the distance between any two of them, find the shortest tour covering all cities.

Definition 4.2 (TSP (decision problem)).

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Output: is there a tour (cycle) visiting every vertex of G with total cost $\leq B$?

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Thank you!