Unique Games Conjecture

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$\mu \prod \lambda \forall$

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The Culprit

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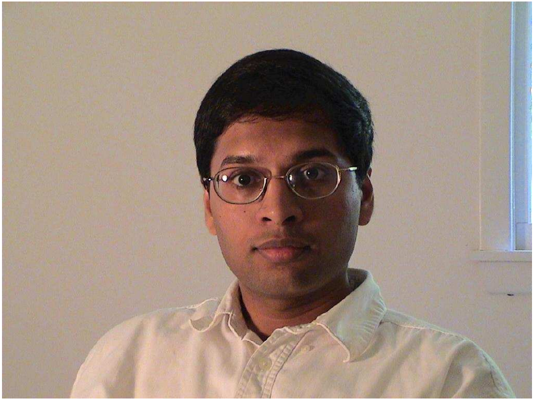

Subhash Khot

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We denote by $\mathsf{VAL}(\mathsf{LC}) \in [0, 1]$ the maximum possible fraction of satisfied edges by any labeling.

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Thus, it is NP-hard to determine if we can satisfy all edges. Ok, so Label Cover is a hard problem!

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  - The labels for x andϕ give the same value to x.

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Notice now that finding a good strategy for the provers is equivalent to solving the label cover problem!

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For every $n > 0$ there exists an L such that it is NP*-hard to distinguish* Label Cover *instances,* LC*, with* $\mathsf{VAL}(\mathsf{LC}) = 1$ *from those with* $\mathsf{VAL}(\mathsf{LC}) \leq n$ *for instances* LC *with the provided label set L.*

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Ok, hold that thought for a second!

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Unique Label Cover Hardness

How hard is Unique Label Cover?

Unique Games Conjecture (UGC)

Conjecture (Khot 2002)

For any $n > 0$ there exists an L such that it is NP*-hard to distinguish* *Unique Label Cover* *instances with* $\mathsf{VAL}(\mathsf{ULC}) > 1 - n$ *from those with* $\mathsf{VAL}(\mathsf{ULC}) \leq n$ *for instances* ULC *with the provided label set L.*

Motivation

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Despite continuous efforts to prove or disprove it, there is still no consensus regarding its validity.

This seems to indicate that UGC is more likely to be resolved in the near future in contrast to the P-NP problem for example, for which it is widely believed that P$\neq$NP but current techniques have not been able to prove it.

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In addition, in recent years, UGC has also proved to be intimately connected to the limitations of Semidefinite Programming (SDP). In particular, if UGC is true, then for every Constraint Satisfaction Problem (CSP) the best approximation ratio is given by a certain simple SDP.

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Notice that under the UGC, UG-hard $\Rightarrow$ NP-hard!

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By using the UGC, Khot and Regev proved in 2003 that it is UG-hard to approximate Min-VC within $2 - \varepsilon$ for any $\varepsilon > 0$

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By using the UGC, Samorodnitsky and Trevisan proved in 2005 that it is UG-hard to approximate Max-IS within $\frac{(\log \Delta)^c}{\Delta}$, for some constant $c > 0$.

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Dinur, Mossel, and Regev proved in 2005 that it is UG-hard to distinguish between $R(G) < \varepsilon|V|$ and $R(G) \geq (1-\varepsilon)|V|$ for any $\varepsilon > 0$.

Maximum Cut (Max-Cut)

We know that Max-Cut is easy to approximate within a factor of $\frac{1}{2}$ (*start with an arbitrary partition of the vertices of the graph and repeatedly move one vertex at a time from one side of the partition to the other, improving the solution at each step, until no more improvements of this type can be made*).

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Khot, Kindler, Mossel and O'Donnell proved in 2004 that it is UG-hard to approximate Max-Cut within $\alpha_{GW} + \varepsilon$, for any $\varepsilon > 0$

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..More Results

Problem	Best Approx. Known	Best Inapprox. Under UGC	Best Inapprox. Known
Max-kCSP	$O(2^k/k)$	$\Omega(2^k/k)$	$2^{k-O(\sqrt{k})}$
Max Acyclic Subgraph	2	$2-\varepsilon$	$\frac{66}{65}-\varepsilon$
Feedback Arc Set	$O(\log N)$	$\omega(1)$	APX-hard
Non-uni. Sparsest Cut	$O(\log N)$	$\omega(1)$	APX-hard
Min-2SAT-Deletion, Min-Uncut	$O(\sqrt{\log N})$	$\omega(1)$	APX-hard
Coloring 3-colorable Graphs	$N{\cdot}2111$	$\omega(1)$	5
Multiway Cut integr. gap $a \leq 1.344$	α	$\alpha-\varepsilon$	APX-hard

Unique Label Cover Inapproximability

As a final note, we state this result by Feige and Reichman in 2004, without its proof.

Theorem (Feige & Reichman 2004)

For any $n > 0$ there exists a $\gamma \in (0, 1)$ and an L such that it is NP*-hard to distinguish between* Unique Label Cover *instances with* $\mathsf{VAL}(\mathsf{ULC}) \geq \gamma$ *and*$\mathsf{VAL}(\mathsf{ULC}) < n\gamma$ *for instances* ULC *with the provided label set L.*

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On the other hand, proving the conjecture incorrect would be an algorithmic breakthrough, likely going beyond the SDP barrier, since the UGC seems to exactly capture the limitations of SDPs.

Irrespective of its truth, the conjecture will probably lead to even more techniques and unconditional results.

References

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Thank you!