Algebraic Computation Models

February 20, 2012

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- Many areas of Computer Science use algebraic computations: computational algebra and geometry, numerical analysis, signal processing, robotics, et.c.
- Useful approximation to the asymptotic behavior of algebraic algorithms, as computers are allowed to use bigger precision day by day (progress in hardware).

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We can avoid such pitfalls by restricting the algorithm's ability to access individual bits.

Algebraic Straight-Line Programs

Something like the following:

Program. Computes $e \times (x_1 + e) + \pi \times x_2$ in $\mathbb{R}$.

Input: x_1, x_2
Output: y_4

$$y_1 = x_1 + e$$
$$y_2 = \pi \times x_2$$
$$y_3 = e \times y_1$$
$$y_4 = y_3 + y_2$$

In the above straight-line program, x_1 and x_2 are given as inputs and the output y_4 is computed from previous y_i's, which are the results of a binary operation in the field; π and e are built-in constants.

- Note: straight-line = no conditionals or loops.

Algebraic Straight-Line Programs (cnt'd)

Definition. An *algebraic straight-line program* of length T with *input variables* $x_1, x_2, \ldots, x_n \in \mathbb{F}$ and *built-in constants* $c_1, c_2, \ldots, c_n \in \mathbb{F}$ is a sequence of T statements of the form

$$y_i = z_{i_1} * z_{i_2},$$

where $*$ is an operation in $\mathbb{F}$, and each of z_{i_1}, z_{i_2} is either an input variable, a built-in constant or y_j for $j < i$.

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The computation consists of executing these simple statements in order, finding values for $y_1, y_2, \ldots, y_T$. The *output* of the computation is the value of y_T.

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Lemma. The output of a straight-line program of length T with variables $x_1, x_2, \ldots, x_n$ is a polynomial $p(x_1, x_2, \ldots, x_n)$ of degree at most 2^T.

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When asking *complexity* questions for a problem computable in this model, we are interested in the length (as a function of n) of the program for an input $x_1, x_2, \ldots, x_n$.

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- FFT: $O(n \log n)$ [Cooley and Tukey].
- Matrix Multiplication: $O(n^3)$ with the naive method and this can be improved using techniques like Strassen's.
- Determinant: $O(n^3)$ using Gaussian elimination.

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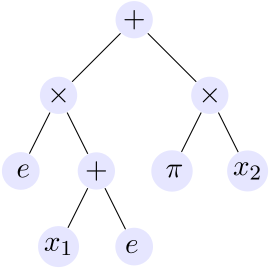
Algebraic Circuits

Again, an example (x_1, x_2 inputs, e, π constants):



The above circuit computes $e \times (x_1 + e) + \pi \times x_2$ in $\mathbb{R}$. Note the similarity with the straight-line program that we saw earlier.

Algebraic Circuits (cnt'd)

Definition. An *algebraic circuit* consists of an acyclic graph. The leaves are called *input nodes*, are labeled $x_1, x_2, \ldots, x_n$ and take values in a field $\mathbb{F}$. We also allow special input nodes labeled with arbitrary constants $c_1, c_2, \ldots, c_k \in \mathbb{F}$. Each internal node, called a *gate*, is labeled with one of the operations $+, \times$. We consider only circuits with a single output node and with the in-degree of each gate being 2.

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The *size* of a circuit is the number of gates in it. The *depth* of the circuit is the length of the longest path from input to output in it.

Lemma. Let $f : \mathbb{F}^n \to \mathbb{F}$ be some function. If f has an algebraic straight-line program of size S, then it has an algebraic circuit of size $3S$. If it is computable by an algebraic circuit of size S then it is computable by an algebraic straight line program of length S.

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Definition. Let $\mathbb{F}$ be a field. We say that a family of polynomials $\{p_n\}_{n \in \mathbb{N}}$, where p_n takes n variables over $\mathbb{F}$, has *polynomially-bounded degree* if there is a constant c s.t. for every n the degree of p_n is at most cn^c.

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Definition. The class $\mathbf{AlgNP_{/poly}}$ is the class of polynomially bounded degree families $\{p_n\}$ that are definable as

$$p_n(x_1, x_2, \ldots, x_n) = \sum_{e\in\{0,1\}^{m-n}} g_m(x_1, x_2, \ldots, x_n, e_{n+1}, \ldots, e_m),$$

where $g_m \in \mathbf{AlgP_{/poly}}$ and m is polynomial in n.

Algebraic Circuits (cnt'd)

Definition. A function $f(x_1, x_2, \ldots, x_n)$ is a *projection* of a function $g(y_1, y_2, \ldots, y_n)$ if there is a mapping σ from $\{y_1, y_2, \ldots, y_n\}$ to $\{0, 1, x_1, x_2, \ldots, x_n\}$ s.t.

$$f(x_1, x_2, \ldots, x_n) = g(\sigma(y_1), \sigma(y_2), \ldots, \sigma(y_n)).$$

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Example. Let $f(x_1, x_2) = x_1 + x_2$; f is projection-reducible to $g(y_1, y_2, y_3) = y_1^2 y_3 + y_2$ since $f(x_1, x_2) = g(1, x_1, x_2)$. ■

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Completeness results based on the above:

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Interesting fact: we need to show $\mathbf{AlgP_{/poly}} \neq \mathbf{AlgNP_{/poly}}$ before we can show $\mathbf{P} \neq \mathbf{NP}$.

Blum-Shub-Smale

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- Generalization of TM's.

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  - Allowing rounding as a basic operation would give it the ability of integer factorization in poly-time.

Blum-Shub-Smale (cnt'd)

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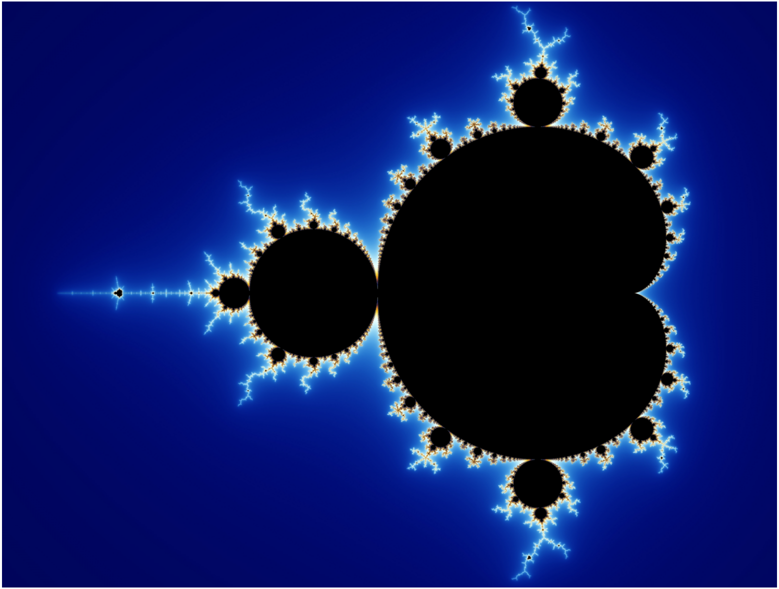
Definition. For complex c, z define $p_c(z) = z^2 + c$. The *Mandelbrot set* is defined as

$$\mathcal{M} = \{c \in \mathbb{C} \mid \text{the sequence } p_c(0), p_c(p_c(0)), \ldots \text{ is bounded } \}.$$



[http://upload.wikimedia.org/wikipedia/commons/2/21/Mandel_zoom_00_mandelbrot_set.jpg]

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Philosophical questions: Roger Penrose vs. Artificial Intelligence.

Bibliography

- S. Arora and B. Barak, Computational complexity: a modern approach, Cambridge University Press, 2009.
- A. Shamir, Factoring numbers in O(log n) arithmetic steps, Inf. Process. Lett., 1979.
- L. G. Valiant, Completeness classes in algebra. In STOC, ACM, 1979.
- L. G. Valiant, The complexity of computing the permanent, Theoretical Computer Science, 1979.
- L. Blum, M. Shub, and S. Smale, On a theory of computation and complexity over the real numbers: NP-completeness, recursive functions and universal machines, American Mathematical Society, 1989.
- R. Penrose, The Emperor's New Mind, Oxford University Press, 1989.