Congestion Games

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Algor. Game Theory

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1 Congestion Games with Player-Specific Payoff Functions
- The model
- The Existence of a Pure-Strategy Nash Equilibrium

2 Congestion Games with Player-Specific Constants
- Congestion Games on Parallel Links
- Arbitrary Congestion Games

(Unweighted) Congestion Games

- The n players share a common set of r strategies.
- The payoff the ith player receives for playing the jth strategy S_{ij} is a monotonically nonincreasing function of the total number of players playing the jth strategy.
- We denote the strategy played by the ith player by σ_i.

The strategy-tuple $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a Nash equilibrium iff each σ_i is a best-reply strategy:

$$S_{i\sigma_i}(n_{\sigma_i}) \geq S_{ij}(n_j + 1)$$

for all i and j.
Here $n_j = \#\{1 \leq i \leq n \mid \sigma_i = j\}$.

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Theorem

Congestion games involving only two strategies possess the *Finite Improvement Property.*

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Lemma

(a) If $j(0), j(1), \ldots, j(M)$ is a sequence of strategies, $\sigma(0), \sigma(1), \ldots, \sigma(M)$ is a best-reply improvement path, and $\sigma(k)$ results from the deviation of one player from $j(k-1)$ to $j(k)$ $(k = 1, 2, \ldots, M)$, then $M \leq n$.

(b) Similarly, if the deviation in the kth step is from $j(k)$ to $j(k-1)$ $(k = 1, 2, \ldots, M)$, then $M \leq n \cdot (r-1)$.

Proof of Theorem

By induction on the number n of players.

- $n = 1$ trivial.
- Assume that the theorem holds for all $(n-1)$-player congestion games.
- We prove it for n-player games.
  - We reduce an n-player congestion game Γ into an $(n-1)$-player game $\bar{\Gamma}$ by "deleting" the last player.
  - $\bar{\Gamma}$ is also a congestion game. The payoff functions $\bar{S}_{ij}$ are defined by

$$\bar{S}_{ij}(\bar{n}_j) = S_{ij}(\bar{n}_j)$$

for $1 \leq i \leq n-1$ and all j, $\bar{n}_j = \#\{1 \leq i \leq n-1 \mid \sigma_i = j\}$.

Proof contd.

- By induction hypothesis, there exists a pure-strategy Nash equilibrium $\bar{\sigma} = (\sigma_1(0), \sigma_2(0), \ldots, \sigma_{n-1}(0))$ for $\bar{\Gamma}$.
- Let $\sigma_n(0)$ be a best reply of player n against $\bar{\sigma}$.
- Starting with $j(0) = \sigma_n(0)$, we can find a sequence $j(0), j(1), \ldots, j(M)$ of strategies and a best-reply improvement path $\sigma(0), \sigma(1), \ldots, \sigma(M)$, as in part (a) of the lemma, such that M is maximal.
- Claim: $\sigma(M) = (\sigma_1(M), \sigma_2(M), \ldots, \sigma_n(M))$ is an equilibrium.

Proof contd.

- Case $\sigma_i(0) \neq \sigma_i(M)$. Strategy $\sigma_i(M)$ is a best-reply against $\sigma(M)$, by the proof of the lemma.
- Case $\sigma_i(0) = \sigma_i(M)$.
  - If $\sigma_i(M) = j(M)$, then $j(M)$ is a best reply against $\sigma(M)$, otherwise there is contradiction to the maximality of M.
  - If $\sigma_i(M) \neq j(M)$, then the number of players playing $\sigma_i(M) = \sigma_i(0)$ is the same in $\sigma(M)$ and $\bar{\sigma}$. Note that $S_{i\sigma_i(0)}(\bar{n}_{\sigma_i(0)}) \geq S_{ij}(\bar{n}_j + 1)$ for all i and j. Also, $n_j(M) \geq \bar{n}_j$ for all j.

Proof contd.

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We conclude that $S_{i\sigma_i(M)}(n_{\sigma_i(M)}(M)) \geq S_{ij}(n_j(M) + 1)$ for all j, and thus $\sigma_i(M)$ is a best reply for i against $\sigma(M)$. □

As a result of the proof of the theorem and the second part of the previous lemma we get the next theorem.

Theorem

Given an arbitrary strategy tuple $\sigma(0)$ in a congestion game Γ, there exists a best-reply improvement path $\sigma(0), \sigma(1), \ldots, \sigma(L)$ such that $\sigma(L)$ is an equilibrium and $L \leq r \cdot \binom{n+1}{2}$.

Some Definitions

A weighted congestion game with player specific constants is a weighted congestion game $\Gamma = (n, E, (w_i)_{i\in[n]}, (S_i)_{i\in[n]}, (f_{ie})_{i\in[n],e\in E})$ with player-specific latency functions such that

(i) for each resource $e \in E$, there is a non-decreasing delay function $g_e : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, and

(ii) for each player $i \in [n]$ and a resource $e \in E$, there is a player-specific constant $c_{ie} > 0$, so that for each player $i \in [n]$ and a resource $e \in E$, $f_{ie} = c_{ie} \cdot g_e$.

- A profile is a tuple $s = (s_1, \ldots, s_n) \in S_1 \times \ldots \times S_n$.
- The load $\delta_e(s)$ for the profile s, on resource $e \in E$ is given by $\delta_e(s) = \sum_{i \in [n] | s_i \ni e} w_i$.
- The Individual Cost of a player $i \in [n]$, for the profile s, is given by $IC_i(s) = \sum_{e \in s_i} f_{ie}(\delta_e(s)) = \sum_{e \in s_i} c_{ie} \cdot g_e(\delta_e(s))$.

↠ In the unweighted case, $w_i = 1$ for all players $i \in [n]$.

Theorem

Every unweighted congestion game with player-specific constants on parallel links has an ordinal potential.

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Proof

We will show that function Φ with

$$\Phi(s) = \prod_{e \in E} \prod_{i=1}^{\delta_e(s)} g_e(i) \cdot \prod_{i=1}^{n} c_{is_i},$$

for any profile s, is an ordinal potential.

Proof (contd.)

- Fix a profile s.
- Consider an improvement step of player $k \in [n]$ to strategy t_k, which transforms s to t.
- We get $IC_k(s) > IC_k(t) \Leftrightarrow g_{s_k}(\delta_{s_k}(s)) \cdot c_{ks_k} > g_{t_k}(\delta_{t_k}(t)) \cdot c_{kt_k}$.
- Function Φ with the new profile becomes

$$\Phi(t) = \Phi(s) \cdot \frac{g_{t_k}(\delta_{t_k}(t)) \cdot c_{kt_k}}{g_{s_k}(\delta_{s_k}(s)) \cdot c_{ks_k}}.$$

We know that the value of the fraction is < 1, because of the improvement step. Hence, $\Phi(t) < \Phi(s)$ and Φ is an ordinal potential. □

Some extra results

Theorem

There is a weighted congestion game with additive player-specific constants and 3 players on 3 parallel links that does not have the *Finite Best-Reply Property*.

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There is a weighted congestion game with additive player-specific constants and 3 players on 3 parallel links that does not have the *Finite Best-Reply Property*.

Proof

By construction!

Theorem

Let Γ be a weighted congestion game with player-specific latency functions and 3 players on parallel links. If Γ does not have a best-reply cycle

$$\langle l, j, j\rangle \rightarrow \langle l, l, j\rangle \rightarrow \langle k, l, j\rangle \rightarrow \langle k, l, l\rangle \rightarrow \langle k, j, l\rangle \rightarrow \langle l, j, l\rangle \rightarrow \langle l, j, j\rangle$$

(where $l \neq j, j \neq k, l \neq k$ are any three links and $w_1 \geq w_2 \geq w_3$) then Γ has a pure Nash equilibrium.

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(where $l \neq j, j \neq k, l \neq k$ are any three links and $w_1 \geq w_2 \geq w_3$) then Γ has a pure Nash equilibrium.

Proof

Going over all the cases!

We now consider weighted congestion games with player-specific affine latency functions where $f_{ie}(x) = a_e \cdot x + c_{ie}$, $i \in [n]$ and $e \in E$.

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Theorem

Every weighted congestion game with player-specific affine latency functions has an ordinal potential.

Proof

We will show that function Φ with

$$\Phi(s) = \sum_{i=1}^{n} \sum_{e \in s_i} w_i \cdot (2 \cdot c_{ie} + a_e \cdot (\delta_e(s) + w_i))$$

for any profile s, is an ordinal potential.

Proof contd.

- Fix a profile s.
- Consider an improvement step of player $k \in [n]$ to strategy t_k, which transforms s to t.
- We get $IC_k(s) > IC_k(t) \Leftrightarrow$
$\sum_{e \in s_k}(a_e \cdot \delta_e(s) + c_{ke}) > \sum_{e \in t_k}(a_e \cdot \delta_e(t) + c_{ke}) \Leftrightarrow$
$\sum_{e \in s_k \setminus t_k}(a_e \cdot \delta_e(s) + c_{ke}) > \sum_{e \in t_k \setminus s_k}(a_e \cdot \delta_e(t) + c_{ke})$.
- Function Φ with the new profile becomes ...

Proof contd.

$$
\begin{aligned}
\Phi(t) = \Phi(s) + \\
& \Big(-\textstyle\sum_{e \in s_k \setminus t_k} w_k \cdot (2 \cdot c_{ke} + a_e \cdot (\delta_e(s) + w_k)) \\
& + \textstyle\sum_{e \in t_k \setminus s_k} w_k \cdot (2 \cdot c_{ke} + a_e \cdot (\delta_e(t) + w_k)) \\
& - \textstyle\sum_{i \in [n] \setminus k} \sum_{e \in s_k \setminus t_k} w_i \cdot a_e \cdot w_k \\
& + \textstyle\sum_{i \in [n] \setminus k} \sum_{e \in t_k \setminus s_k} w_i \cdot a_e \cdot w_k \Big) \Leftrightarrow \ldots
\end{aligned}
$$

Proof contd.

$$\Phi(t) = \Phi(s) + \Big(-\sum_{e \in s_k \setminus t_k} w_k \cdot (2 \cdot c_{ke} + a_e \cdot (\delta_e(s) + w_k)) + \sum_{e \in t_k \setminus s_k} w_k \cdot (2 \cdot c_{ke} + a_e \cdot (\delta_e(t) + w_k)) - w_k \cdot \sum_{e \in s_k \setminus t_k} a_e \cdot (\delta_e(s) - w_k) + w_k \cdot \sum_{e \in t_k \setminus s_k} a_e \cdot (\delta_e(t) - w_k)\Big) \Leftrightarrow \dots$$

Proof contd.

$$\Phi(t) = \Phi(s) + \Big(-2 \cdot w_k \cdot \textstyle\sum_{e \in s_k \setminus t_k} c_{ke} + a_e \cdot \delta_e(s) + 2 \cdot w_k \cdot \textstyle\sum_{e \in t_k \setminus s_k} c_{ke} + a_e \cdot \delta_e(t) \Big)$$

We know that the value of the parenthesis is < 0, because of the impovement step. Hence, $\Phi(t) < \Phi(s)$ and Φ is an ordinal potential. □