NETWORK DESIGN AND THE BRAESS PARADOX

Algorithmic Game Theory

Corelab
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Outline

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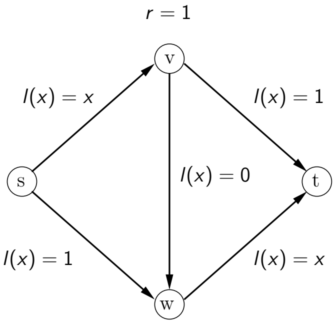
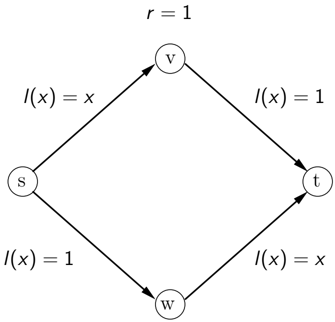
Selfish Routing

- Problem: route traffic in a network of selfish non-cooperative players.
- Motivation: simple examples show that Nash equilibria can be inefficient (Price of Anarchy).
- Question: which subnetwork will exhibit the best performance when used selfishly?

Braess's Paradox





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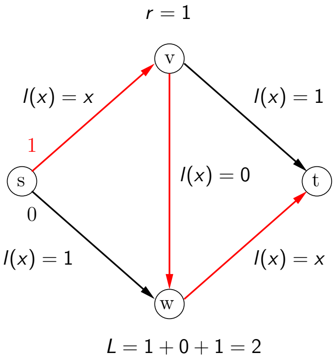

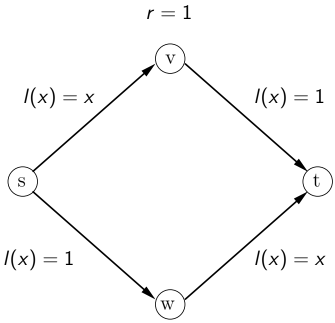

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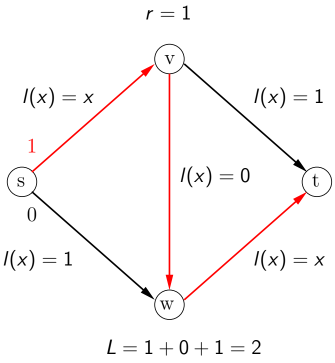

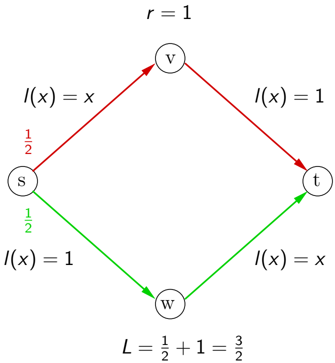

The model

- Directed network $G = (V, E)$.
- Source s and destination t.
- Latency function $l_e : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$. We assume that l_e's are continuous and non-decreasing.
- Traffic r, caused by an infinite population of players, each trying to route a negligible amount of traffic through an $s - t$ path.
- Let $\mathcal{P}$ be the set of simple $s - t$ paths. Then, a flow f is non-negative vector indexed by $\mathcal{P}$.
- Feasible flow f: $\sum_{P \in \mathcal{P}} f_P = r$.
- Flow on edges $f_e = \sum_{P : e \in P} f_P$.
- Latency of a path P: $l_P(f) = \sum_{e \in P} l_e(f_e)$.
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Intuitively, each unit of flow travels along the minimum-latency path in a NE.

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A flow f feasible for (G, r, l) is at Nash equilibrium, or is a Nash (or Wardrop) flow, if for all $P_1, P_2 \in \mathcal{P}$ with $f_{P_1} > 0$ and $\delta \in (0, f_{P1}]$, we have

$$l_{P_1}(f) \leq l_{P_2}(\tilde{f}),$$

where

$$\tilde{f}_P = \begin{cases} f_P - \delta, & \text{if } P = P_1 \\ f_P + \delta, & \text{if } P = P_2 \\ f_P, & \text{otherwise} \end{cases}$$

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Thus, all $s - t$ paths in NE with positive flow have **equal** latency, denoted by $L(G, r, l)$.

Moreover, flows at NE always exist and are **unique** with respect to $L(G, r, l)$.

Finally, there exists a directed acyclic Nash flow.

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Formalizing our Problem

Problem

Given an instance (G, r, l), find a subgraph H of G that minimizes $L(H, r, l)$.

Properties of Nash Flows

Lemma

For every instance (G, r, l), $L(G, r, l)$ is a non-decreasing function of r.

Lemma

Let f be a flow feasible for (G, r, l). For a vertex v in G, let $d(v)$ denote the length, with respect to edge lengths $\{l_e(f_e)\}_{e \in E}$ of a shortest $s - v$ path in G. Then f is at Nash equilibrium iff

$$d(w) - d(v) \leq l_e(f_e)$$

for all edges $e = (v, w)$, with equality holding whenever $f_e > 0$.

Lemma

If f is a flow at NE for (G, r, l), then $C(f) = r \cdot L(G, r, l)$.

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Linear Latency Functions

We consider latency functions of the form $l_e(x) = a_e x + b_e$, $a_e, b_e \geq 0$. We then call the problem the LINEAR LATENCY NETWORK DESIGN. It is known that the price of anarchy in such networks is at most $\frac{4}{3}$.

Algorithm (Trivial Algorithm)

Given an instance (G, r, l), build the whole network G.

Lemma (Roughgarden - Tardos)

Let f^* and f be feasible and Nash flows, respectively, for an instance (G, r, l) with linear latency functions. Then,

$$C(f) \leq \frac{4}{3} \cdot C(f^*).$$

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The trivial algorithm performance for LINEAR LATENCY NETWORK DESIGN

Corollary

The trivial algorithm is a $\frac{4}{3}$-approximation algorithm for LINEAR LATENCY NETWORK DESIGN.

Proof.

- Let H be the subgraph that minimizes $L(H, r, l)$, and f and f^* be the flows at NE for (G, r, l) and (H, r, l).
- $C(f) = r \cdot L(G, r, l)$ and $C(f^*) = r \cdot L(H, r, l)$.
- f^* feasible for (G, r, l), thus $C(f) \leq \frac{4}{3} C(f^*)$.
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Optimality of the trivial algorithm (1 / 3)

Theorem

For every $\epsilon > 0$, there is no $\left(\frac{4}{3} - \epsilon\right)$-approximation algorithm for LINEAR LATENCY NETWORK DESIGN, assuming $\mathrm{P} \neq \mathrm{NP}$.

We will use a reduction from the 2 DIRECTED DISJOINT PATHS (2DDP) problem: given a directed graph $G = (V, E)$ and distinct vertices $s_1, s_2, t_1, t_2 \in V$, are there $s_i - t_i$ paths P_i for $i = 1, 2$, such that P_1 and P_2 are vertex-disjoint?

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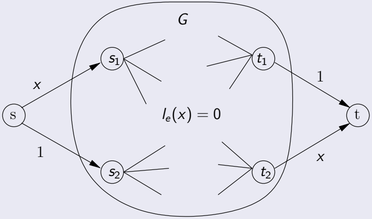

$(G', 1, l)$

- If algorithm returns a subgraph H with $L(H, 1, l) < 2$, then "yes" instance of 2DDP, else "no".
- If "yes" instance, let P_1 and P_2 be vertext disjoint $s_1 - t_1$ and $s_2 - t_2$ paths. Obtain H by deleting all other edges. Observe now that $L(H, 1, l) = \frac{3}{2}$ (1/2 routed in $s_1 \to t_1 \to t$ and 1/2 in $s_2 \to t_2 \to t$). So, $ALG \leq \left(\frac{4}{3} - \epsilon\right) \cdot \frac{3}{2} < 2$.

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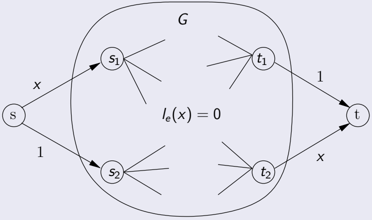

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Optimality of the trivial algorithm (3 / 3)

Proof (continued).

- We will prove that if "no" instance, then $L(H, 1, l) \geq 2$ for all subgraphs of G', and so $ALG \geq 2$.
- Split subgraphs of G' in 3 groups: (i) those with an $s_2 - t_1$ path, (ii) those with an $s_1 - t_2$ path and (iii) those with an $s_i - t_i$ path for exactly one $i \in \{1, 2\}$.
- In all cases, routing flow in such a path gives NE and $L(H) = 2$.
- Thus, $ALG \geq OPT \geq 2$, and so we solve 2DDP.

□

Interpretation of results

- Efficiently detecting Braess's Paradox in networks with linear latency functions is impossible (i.e. NP-hard). This holds even in the most severe cases, where $PoA = \frac{4}{3}$.
- However, by restricting our linear latency functions only to strictly increasing ones, we can get positive results!

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Towards some positive results

For instances with strictly increasing linear latencies, the optimal flow is **unique** and can be efficiently computed.

Definition

An instance (G, r, l) is called *paradox-free* if for every subgraph H of G, $L(H, r, l) \geq L(G, r, l)$. An instance (G, r, l) is called *paradox-ridden* if there is a subgraph H of G, such that $L(H, r, l) = L^*(G, r, l) = L(G, r, l)/PoA(G, r, l) \leq L(G, r, l)$.

Note: In a paradox-free instance PoA cannot be improved by edge removal.

Lemma

An instance (G, r, l) with $G = (V, E)$ is paradox-ridden iff there is an optimal flow f^* that is a Nash flow on the subgraph $G^*(V, E^*)$, where $E^* = \{e \in E : f^*_e > 0\}$.

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For instances with strictly increasing linear latencies, the optimal flow is **unique** and can be efficiently computed.

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An instance (G, r, l) is called *paradox-free* if for every subgraph H of G, $L(H, r, l) \geq L(G, r, l)$. An instance (G, r, l) is called *paradox-ridden* if there is a subgraph H of G, such that $L(H, r, l) = L^*(G, r, l) = L(G, r, l)/PoA(G, r, l) \leq L(G, r, l)$.

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Detecting paradox-ridden networks

Theorem (Fotakis, Kaporis, Spirakis)

Given an instance (G, r, l) with strictly increasing linear latencies, one can decide in polynomial time whether the instance is paradox-ridden or not.

Proof.

- We can efficiently compute the *unique* optimal flow f^*.
- We then compute the length $d(v)$ of a shortest $s - v$ path wrt edge lengths $\{l_e(f_e^*)\}_{e \in E^*}$ for all $v \in V$.
- f^* Nash flow $\Leftrightarrow \forall (u, v) \in E^*, d(v) = d(u) + l_{(u,v)}(f^*_{(u,v)})$.

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Towards a positive result for arbitrary linear latencies

- As already stated, we cannot decide whether an instance with arbitrary linear latencies is paradox-ridden or not.
- However, we can reach sufficient conditions under which we can answer the above question.
- Let (G, r, l) be an instance with $l_e(x) = a_e(x) + b_e$ and $E^c = \{e \in E : a_e = 0\}$. Let $E^i = E \setminus E^c$ and let $\mathcal{O}$ be the set of optimal flows.

Note: All optimal flows assign the same traffic to the edges with strictly increasing latencies, and can differ only on edges with constant latencies.

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An LP formulation

(LP):

$$\min \quad \sum_{e \in E^c} f_e b_e, \quad s.t.:$$

$$\sum_{u:(v,u)\in E^i} o_{(v,u)} + \sum_{u:(v,u)\in E^c} f_{(v,u)} = \sum_{u:(u,v)\in E^i} o_{(u,v)} + \sum_{u:(u,v)\in E^c} f_{(u,v)} \qquad \forall v \in V \setminus \{s,t\},$$

$$\sum_{u:(s,u)\in E^i} o_{(s,u)} + \sum_{u:(s,u)\in E^c} f_{(s,u)} = r,$$

$$\sum_{u:(u,t)\in E^i} o_{(u,t)} + \sum_{u:(u,t)\in E^c} f_{(u,t)} = r,$$

$$f_e \geq 0 \qquad \forall e \in E^c.$$

Some notes on the (LP)

- An optimal solution to (LP) corresponds to a feasible flow for (G, r, l) that agrees with o on all edges in E^i and allocates traffic to the edges in E^c so that the total latency is minimized.
- Optimal solutions to (LP) $\xleftrightarrow{1-1}$ Optimal flows in $\mathcal{O}$.
- Given an optimal flow o, the problem of checking if there is a $o \in \mathcal{O}$ that is a Nash flow on G_o reduces to the problem of generating all optimal solutions of (LP) and checking whether some of them can be translated into a Nash flow on the corresponding subnetwork.
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A positive result for arbitrary linear latencies (1 / 2)

Theorem

Given an instance (G, r, l) with arbitrary linear latencies where the corresponding (LP) has a unique optimal solution, one can decide in polynomial time whether the instance is paradox-ridden or not.

Note: In fact, it suffices to generate all optimal basic feasible solutions, as the (LP) allocates traffic to constant latency edges. Observe that if a feasible flow f is a Nash flow, then any solution f' with $\{e : f'_e > 0\} \subseteq \{e : f_e > 0\}$ is a Nash flow, too.

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Given an instance (G, r, l) with arbitrary linear latencies where the corresponding (LP) has a polynomial number of basic feasible solutions, one can decide in polynomial time whether the instance is paradox-ridden or not.

Note: The above class includes instances with a constant number of constant latency edges.

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Finding near-optimal subnetworks

- In general, finding optimal subnetworks in paradox-ridden instances is NP-hard.
- However, we can reach a subexponential-time approximation scheme on networks with polynomially many paths, each of polylogarithmic length.
- For this purpose, we need to turn our attention to "sparse" flows and ε-Nash flows.

Definition (ε-Nash flow)

For some $\varepsilon > 0$, a flow f is an ε-Nash flow if for every path P and P' with $f_P > 0$, $l_P(f) \leq l_{P'}(f) + \varepsilon$.

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Making a flow "sparse" (1 / 3)

Lemma (Fotakis, Kaporis, Spirakis)

Let $(G, 1, l)$ be an instance on a graph $G = (V, E)$, and let f be any feasible flow. For any $\varepsilon > 0$, there exists a feasible flow $\tilde{f}$ that assigns positive traffic to at most $\lfloor \log(2m)/(2\varepsilon^2) \rfloor + 1$ paths, such that $|\tilde{f}_e - f_e| \leq \varepsilon$, $\forall e \in E$.

Proof.

- Let $\mu = |\mathcal{P}|$, and we index the $s - t$ paths by integers in $[\mu]$.
- Flow f can be seen as a probability distribution on $\mathcal{P}$.
- We prove that if we select $k > \log(2m)/(2\varepsilon^2)$ paths uniformly at random with replacement according to f, and assign to each path j a flow equal to the number of times j is selected divided by k, we obtain a flow that is an ε-approximation to f with positive probability. (Probabilistic Method)

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Making a flow "sparse" (2 / 3)

Proof (continued).

- Fix ε and let $k = \lfloor \log(2m)/(2\varepsilon^2) \rfloor + 1$.
- Define random variables $P_1, ...P_k \in [\mu]$, i.i.d., such that $\mathrm{P}[P_i = j] = f_j$.
- For each path $j \in [\mu]$, $F_j = |\{i \in [k] : P_i = j\}| / k$. Note that $\mathbf{E}[F_j] = f_j$.
- For each edge e and random variable P_i, define the independent indicator variables $F_{e,i} = 1$ if e in path P_i, otherwise 0.
- Let $F_e = \frac{1}{k} \sum_{i=1}^{k} F_{e,i}$. Observe that $F_e = \sum_{j:e \in j} F_j$ and $\mathbf{E}[F_e] = f_e$.

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Making a flow "sparse" (3 / 3)

Proof (continued).

- Note that $\sum_{j=1}^{\mu} F_j = 1$. Thus, $F_1, \ldots, F_\mu$ define a feasible flow that assignes positive traffic to at most k paths and "agrees" with f on expectation.
- By the Chernoff-Hoeffding bound we get that for every edge e

$$\mathrm{P}[|F_e - f_e| > \varepsilon] \leq 2e^{-2\varepsilon^2 k} < 1/m$$

- Thus, by union bound, $\mathrm{P}[\exists e : |F_e - f_e| > \varepsilon] < m(1/m) = 1$.
- So, there is positive probability that the flow $(F_1, \ldots, F_\mu)$ satisfies $|F_e - f_e| \leq \varepsilon$, $\forall e \in E$. Thus, there exists a flow $\tilde{f}$ with the properties of $(F_1, \ldots, F_\mu)$.

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Finding a near-optimal subnetwork

Theorem

Let $(G(V,E), 1, \{a_e x + b_e\}_{e \in E})$ *be an instance,* $\alpha = \max_{e \in E}\{a_e\}$*, and let* H^B *be the best subnetwork of* G*. For some constants* $d_1, d_2 > 0$*, let* $|\mathcal{P}| \leq m^{d_1}$ *and* $|P| \leq \log^{d_2} m$*, for all* $P \in \mathcal{P}$*. Then, for any* $\varepsilon > 0$*, we can compute in time* $m^{O(d_1 \alpha^2 \log^{2d_2+1}(2m)/\varepsilon^2)}$ *a flow* $\tilde{f}$ *that is an* ε*-Nash flow on* $G_{\tilde{f}}$ *and satisfies* $l_P(\tilde{f}) \leq L(H^B, 1, \{a_e x + b_e\}_{e \in E(H^B)}) + \varepsilon/2$*, for all paths* $P \in G_{\tilde{f}}$*.*

Moving on to general latency functions

- We will now consider general (continuous, non-decreasing) latency functions (we call this problem the GENERAL LATENCY NETWORK DESIGN).
- We will see that the trivial algorithm is still the best thing we can do. However, the approximation factor gets worse.
- In order to prove the above, we will need new techniques, as in networks with general latency functions, a Nash flow can be arbitrarily more costly than other feasible flows.

A useful tool in our approach

Definition

Let f be a Nash flow for the instance (G, r, l). Let $d(v)$ denote the length of a shortest $s - v$ path with respect to the edge lengths $\{l_e(f_e)\}_{e \in E}$. An ordering of the vertices of G is f-monotone if it satisfies the following two properties:

(P1) All f-flow travels forward in the ordering.

(P2) The d-values of vertices are non-decreasing in the ordering.

Note: It can be proved that an f-monotone ordering exists relative to a directed acyclic Nash flow.

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The trivial algorithm performance for GENERAL LATENCY NETWORK DESIGN

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The trivial algorithm is a $\lfloor n/2 \rfloor$-approximation algorithm for GENERAL LATENCY NETWORK DESIGN.

Proof.

On board

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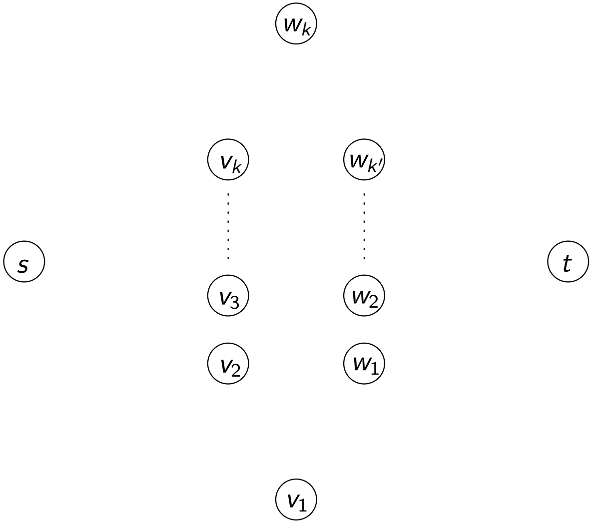
Tightness of the $\lfloor n/2 \rfloor$ bound: the B^k Braess Graph



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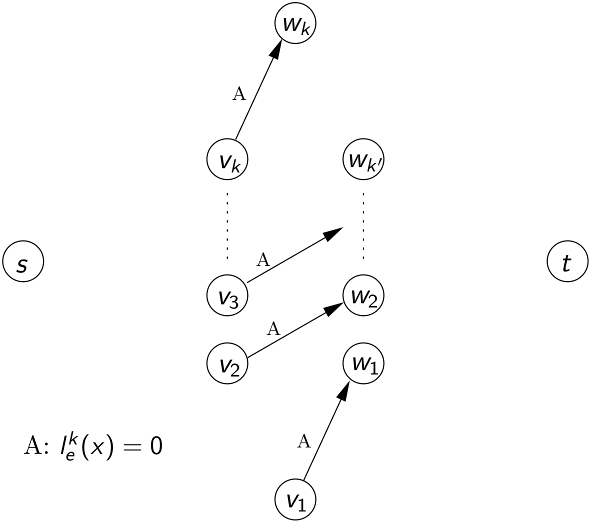

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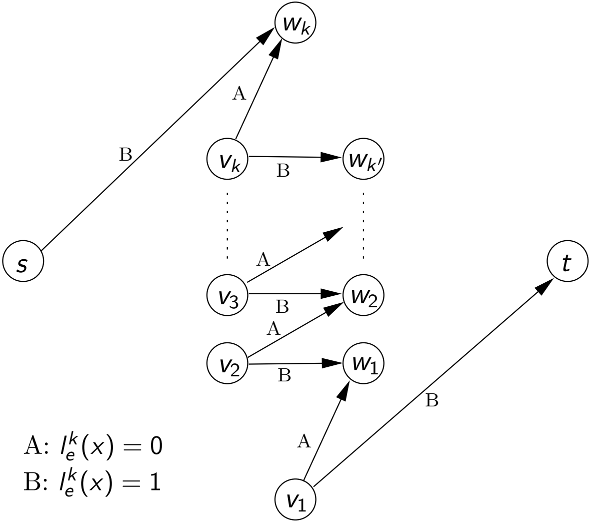

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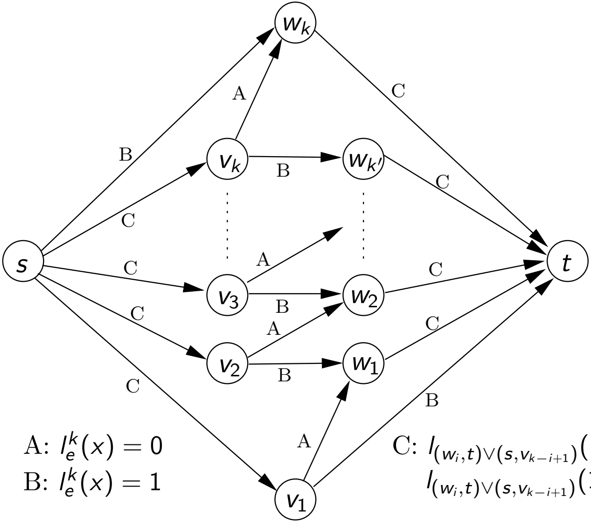
B: $l_e^k(x) = 1$

Tightness of the $\lfloor n/2 \rfloor$ bound: the B^k Braess Graph



A: $l_e^k(x) = 0$
B: $l_e^k(x) = 1$

C: $l_{(w_i,t)\vee(s,v_{k-i+1})}(k/(k+1)) = 0$
$l_{(w_i,t)\vee(s,v_{k-i+1})}(1) = i$

Tightness of the $\lfloor n/2 \rfloor$ bound (1 / 2)

Theorem

For every integer $n \geq 2$, there is an instance (G, r, l) in which G has n vertices and a subgraph H with

$$L(G, r, l) = \left\lfloor \frac{n}{2} \right\rfloor \cdot L(H, r, l).$$

Proof.

- Assume that $n \geq 4$ is even (otherwise, add an isolated vertex).
- So, $n = 2k + 2$ and we consider the instance (B^k, k, l^k).
- NE for (B^k, k, l^k): 1 unit on each path $s \to v_i \to w_i \to t$, and $L(B^k, k, l^k) = k + 1$.

Tightness of the $\lfloor n/2 \rfloor$ bound (2 / 2)

Proof (continued).

- We now remove all A-type edges and obtain H.
- Routing $k/(k+1)$ units on paths $s \rightarrow v_1 \rightarrow t$, $s \rightarrow w_k \rightarrow t$ and $\{s \rightarrow v_i \rightarrow w_{i-1} \rightarrow t\}_{(i=2,\ldots,k)}$, we get a NE with $L(H, k, l^k) = 1$.
- Thus, $L(G)/L(H) = k + 1 = n/2$.

□

Hardness of approximation for GENERAL LATENCY NETWORK DESIGN

Theorem (Roughgarden)

For every $\epsilon > 0$, there is no $(\lfloor n/2 \rfloor - \epsilon)$-approximation algorithm for GENERAL LATENCY NETWORK DESIGN, assuming $\mathrm{P} \neq \mathrm{NP}$.

Proof is based on a reduction from the NP-complete problem PARTITION.

Price of anarchy in networks with general latency functions

Theorem (Lin, Roughgarden, Tardos)

For every $n \geq 2$ and every single-commodity instance (G, r, l) with n vertices, $PoA(G, r, l) \leq n - 1$.

Lemma

For all $k \geq 1$, the only way to decrease the latency in a Nash flow by a factor strictly larger than k is to remove at least k edges from the network.

Theorem

The worst-case price of anarchy in multicommodity instances with at most n vertices is $2^{\Omega(n)}$ as $n \to \infty$. Moreover, there are instances in which PoA can be reduced to 1 by edge removal.

Outline

1. Introduction
2. Approximation Algorithms - Inapproximability results
3. Frequency of Braess's Paradox

How often does Braess's paradox occur?

Question: Is Braess's paradox often in practical networks or is it just a theoretical curiosity?

Valiant and Roughgarden answer that it occurs in many networks by utilizing random graph models.

Definition (Braess ratio)

The Braess ratio of a network is the largest factor by which the removal of one or more edges can improve the latency of traffic in an equilibrium flow.

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The model

- Probability distribution oven graphs and edge latency functions.
- Graph G distributed according to the standard Erdös-Renyi $\mathcal{G}(n, p)$ model. For a fixed $n \geq 2$, each edge is present independently with probability p. We assume that $p = \Omega(n^{-1/2+\epsilon})$ for some $\epsilon > 0$.
- Source s and destination t are chosen randomly or arbitrarily. (we assume that there is no edge (s, t)).
- Linear latency functions $l(x) = ax + b$, $a, b \geq 0$:
  1. *Independent coefficients* model: two fixed distributions $\mathcal{A}$ and $\mathcal{B}$, and each edge is independently given a latency function $l(x) = ax + b$, where a and b are drawn independently from $\mathcal{A}$ and $\mathcal{B}$, respectively.
  2. 1/x model: each edge present in the graph (independently) has the latency function $l(x) = x$ with probability q and $l(x) = 1$ with probability $1 - q$.

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Main results

Theorem (Independent coefficients model)

Let $\mathcal{A}$ and $\mathcal{B}$ be reasonable distributions. There is a constant $p = p(\mathcal{A}, \mathcal{B}) > 1$ such that, with high probability, a random network (G, l) admits a choice of traffic rate r such that the Braess ration of the instance (G, r, l) is at least p.

Theorem (The 1/x model)

There is a traffic rate $R = R(n, p, q)$ such that, with high probability as $n \to \infty$, the Braess ratio of a random n-node network from $\mathcal{G}(n, p, q)$ with traffic rate R is at least

$$\frac{4 - 3pq}{3 - 2pq}.$$

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THANK YOU!