PLS-Completeness of computing a Pure Nash Equilibrium

A. Antonopoulos (N.T.U.A.)

T.C.S. II: Algorithmic Game Theory

5/5/2011

Local Search Problems

Definition

A *local search problem* Π is given by its set of instances $\mathcal{I}_\Pi$:

- For every $I \in \mathcal{I}_\Pi$, we have a set of feasible solutions $\mathcal{F}(I)$ and an objective function $c : \mathcal{F}(I) \rightarrow \mathbb{N}$.
- For every $S \in \mathcal{F}(I)$, we have a neighborhood $\mathcal{N}(S, I) \subseteq \mathcal{F}(I)$

Given an I, we seek for a **locally optimal** solution S^*:
$c(S) \substack{\geq \\ \leq} c(S^*), \forall S \in \mathcal{N}(S^*, I)$ (a solution with <u>no</u> better neighbor).

Reminder

- A *Pure Nash Equilibrium* (PNE) is a state $s = (s_1, \cdots, s_n)$ s.t. for each i:
  $u_i(s_1, \ldots, s_i, \ldots, s_n) \geq u_i(s_1, \ldots, s_i', \ldots, s_n), \forall s_i' \in S_i$.
- A game is *symmetric* of all $S_i's$ are the same and all u_i's are identical symmetric functions of $n - 1$ variables.

Congestion Games

- Consider a graph with:
  - Nodes: $S_1 \times S_n$
  - Edges: (s, s'), if s and s' differ only in one component (the i^{th}) and $u_i(s') > u_i(s)$
- If this graph is **acyclic**, then there is a pure Nash Equilibrium!

Congestion Games

A Congestion Game $M = (N, E, (S_i)_{i \in \mathbb{N}}, d)$ is a tuple:

- N a set of players
- E a set of resources
- S_i action sets, with $S_i \subseteq 2^E$
- Delay function $d : E \times \{0, 1, \ldots, n\} \to \mathbb{N}$ (denoted as $d_e(j)$, nondecreasing in j)

Congestion Games

- Payoffs:
  If $s = (s_1, s_2, \ldots, s_n)$ a state, let $f_s(e) = |\{i : e \in s_i\}|$. Then:

$$c_i(s) = \sum_{e \in s_i} d_e(f_s(e))$$

Theorem

Every Congestion Game has a Pure Nash Equilibrium.

- In a network congestion game the families of S_i are presented **implicitly** as paths in a network:
  Given (V, E), nodes a_i, b_i for each player i and a delay function, Edges are palying the role of *resources*, and the subset of E available to player i is the set of all paths from a_i to b_i.

The Class PLS

- The class **PLS** contains all local search problems with polynomial-time searchable neighborhoods:

Definition

A local search problem Π belongs to class **PLS** if there exists polynomial-time algorihtms for the following:

- An algorithm A that computes for every I of Π an initial *feasible* solution $S^0 \in \mathcal{F}(I)$.
- An algorithm B that computes for every I of Π and every $S \in \mathcal{F}(I)$ the objective value $c(S)$.
- An algorithm C that determines $\forall I \in \mathcal{I}_\Pi, \forall S \in \mathcal{F}(I)$ whether S is *locally optimal*, and if not finds a better solution in $\mathcal{N}(S, I)$.

The Class PLS

- **FP** $\subseteq$ **PLS** $\subseteq$ **TFNP**
- We now need a notion of reduction:

Definition

A problem Π_1 in **PLS** is **PLS**-reducible to a problem Π_2 in **PLS**, if there exist polynomial-time computable functions f and g such that:

1. Function f maps instances I of Π_1 to instances $f(I)$ of Π_2.
2. Function g maps pairs (S_2, I) (S_2 is a solution of $f(I)$) to solutions S_1 of I.
3. For all instances I of Π_1 and all solutions S_2 of $f(I)$:
   If S_2 is a *local optimum* of instance $f(I)$, then $g(S_2, I)$ is a *local optimum* of I.

Complexity of Congestion Games

- A network potential game is *symmetric* if all players have the same endpoints a and b.

Theorem

There is a polynomial-time algorithm for finding a Pure Nash Equilibrium in symmetric network congestion games.

Theorem

It is **PLS**-complete to find a Pure Nash Equilibrium in network congestion games of the following sorts:

1. *General* congestion games.
2. *Symmetric* congestion games.
3. *Assymetric network* congestion games.

Complexity of Congestion Games

- The proof for (1) is based on a reduction from the (known to be) **PLS**-complete `POSNAE3FLIP`:

Definition (`POSNAE3FLIP`)

Given an instance of not-all-equal-3SAT with weights on its clauses and containing positive literals only, find a truth assignment satisfying clauses whose total weight cannot be improved by flipping a variable.

So,

- For each 3-clause c of weight w, we have e_c and e'_c, with delay:
    - 0, if there are 2 or fewer players
    - w, otherwise
- Players are variables!

Complexity of Congestion Games

- Player x has 2 strategies:
  - one containing all e_c's for clauses that contain x
  - one containing all e'_c's for the same clauses
- Any Nash Equilibrium of the congestion game is a **local optimum** of the POSNAE3FLIP instance! $\square$

- The proof of (2) is a reduction of the non-symmetric to the symmetric case:
- Let $S'_i = \{s \cup \{e_i\} : s \in S_i\} \forall i$
- e_i's are distinct new resources with $d_{e_i}(j) = 0$, if $j = 1$, and $d_{e_i}(j) = M$, if $j \geq 2$
- Consider the symmetric game with the same edges, and *common* strategy set $\bigcup_i S'_i$.
- Any equilibrium will have 1 player using S'_i, and hence will correspond to (by omitting the e_i's) a specific equilibium of the original game. $\square$

Non-Atomic Congestion Games

- An non-atomic congestion game is the limit of a congestion game as $n \to \infty$.
- We are given a network (V, E) and *endpoint pairs* (a_i, b_i), $i = 1, \ldots, k$, and *flow requirements* r_i (rationals adding to 1)
- For each edge we have a (non-decreasing) *delay function* $d_e : [0, 1] \to \mathbb{R}^+$.
- For a path p and a flow f: $d_p(f) = \sum_{e \in p} d_e(f)$.
- We want to find a k-commodity flow f that is a Nash Eqilibrium, that is, any flow between a_i and b_i (for all pairs a_i,b_i) has a delay *at no larger* than any other a_i-b_i path p'.
- This problem can be rephrased as a convex optimization problem, and so it can be solved by the Ellipsoid algorithm.

Non-Atomic Congestion Games

- We say that a state $s = (s_1, \ldots, s_n$ is an ϵ-approximate Nash Equilibrium if $\forall$ i, every flow path p carrying at least ϵ units of flow and avery $a_i - b_i$ path p', the delay $d_p(f)$ is no larger than $d_{p'}(f) - \epsilon$ (no player has a defection thet decreases his delay more than ϵ).
- By making a Lipschitz assumption for the latency functions d_e: There exists a constant C, such that, $\forall x, y : 0 \leq x < y \leq 1$:

$$|d_e(y) - d_e(x)| \leq C|y - x|$$

we have the following result:

Theorem

Given a non-atomic congestion game with delay functions satisfying the Lipschitz assumption with constant C, an ϵ-approximate Nash Equilibrium can be computed in time $poly(|E|, C, \epsilon^1)$

Potential Games

Definition

A game is called *Exact Potential Game* if there is a function ϕ, s.t. for any edge of the Nash Dynamic Graph (s, s') with defector i we have $\phi(s') - \phi(s) = u_i(s') - u_i(s)$.

Theorem

Any exact potential game is isomorphic to a congestion game.

- The class of general potential games essentially comprises all of **PLS**:

Potential Games

Theorem

For any problem in **PLS** with instances I there is a family of *general potential* games (indexed by I) such that, for instance x, the game G_x has $poly(|x|)$ players each with strategy set that includes the alphabet Σ, and such that the set of Pure Nash Equilibria of G_x is precicely the *local optima* of x.

Matroids

Definition

A tuple $M = (\mathcal{R}, \mathcal{I})$ is a *matroid* if: $\mathcal{R}$ is a finite set of resources, and $\mathcal{I}$ is a (nonempty) family of subsets of $\mathcal{R}$ s.t.:
-If $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$, and
-If $I, J \in \mathcal{I}$ and $|J| < |I|$, then $\exists i \in I \setminus J : J \cup \{i\} \in \mathcal{I}$.

- Let $I \subseteq \mathcal{R}$. If $I \in \mathcal{I}$, then we call I an **indepedent set** or $\mathcal{R}$.
- All maximal indepedent sets of $\mathcal{I}$ have the same size, denoted by $rk(M)$ of the matroid M.
- A maximal indepedent set B is called a basis of M.
- We call a matroid weighted if there is a function $w : \mathcal{R} \rightarrow \mathbb{N}$.
- We want to find a basis of a minimum weight, where the weight of an indepedent set is given by:

$$w(I) = \sum_{r \in I} w(r)$$

Matroids

- Such a basis can be found by a greedy algorithm.

Defintion

A congestion game $\Gamma = (N, \mathcal{R}, (S_i)_{i\in N}, (d_r)_{r\in\mathcal{R}})$ is called a *matroid congestion game* if for every player $i \in N$, $M_i = (\mathcal{R}, \mathcal{I}_i)$ with $\mathcal{I}_i = \{I \subseteq S | S \in S_i\}$ is a matroid and S_i is a set of bases of M_i. Additionally, we denote by

$$rk(\Gamma) = \max_{i\in N} rk(M_i)$$

the rank of the matroid congestion game Γ.

Matroids

- The main result is the following:

Theorem

Let Γ be a matroid congestion game. Then, players reach a Nash Equilibrium after at most $n^2 m \cdot rk(\Gamma)$ best responses. In the case of identical delay functions, players reach a Nash Equilibrium after at most $n^2 \cdot rk(\Gamma)$ best responses.

- The matroid property is a sufficient (and necessary) condition on the combinatorial structure of the players' strategy spaces guaranteeing fast convergence to Nash Equilibria!
- The length of all best responce sequences are polynomially bounded in the number of players and resources.

References

- Alex Fabrikant, Christos Papadimitriou, Kunal Talwar, **The complexity of Pure Nash equilibria**, on Proceedings of the thirty-sixth annual ACM symposium on Theory of computing, 2004
- Heiner Ackermann, Heiko Röglin, Berthold Vöcking, **On the impact of combinatorial structure on congestion games** J. ACM 55(6), 2008
- David S. Johnson, Christos H. Papadimtriou, Mihalis Yannakakis, **How easy is local search?** J. Comput. Syst. Sci., 37:79-100, August 1988.
- Christos H. Papadimitriou, **On the complexity of the parity argument and other inefficient proofs of existence** J. Comput. Syst.Sci., 48:498-532, June 1994.

Thank You!