On strategy-proofness and single peakedness

H. MOULIN*
University Paris-IX

Introduction

The literature on strategic manipulation of decision schemes, following the seminal Gibbard-Satterthwaite theorem [6], must deal with the central negative result stated by this theorem. If the preference preorderings of the agents on the set of alternatives can be any ordering (a condition usually referred to as 'unrestricted domain'), then, apart from the dictatorial ones, every decision scheme will include an incentive for strategic misrepresentation of preferences for at least one preference profile.

Several relaxations of the unrestricted domain assumption have been investigated. Following are descriptions of three of them, the essential ones: in the most recent one, Gibbard proposes to randomize the choice of the elected alternative (thus allowing voting schemes where ties are broken off by flipping a coin) and at the same time assumes that each agent has a Von Neumann-Morgenstern utility to estimate lotteries (the agents are risk-neutral). In this context he proves [7, 8] that non dictatorial strategy-proof voting schemes do exist, but unfortunately are not efficient. A second relaxation of the unrestricted domain condition amounts to assuming that side-payments are allowed among the agents (implying the cardinality of their utility functions): this creates several strategy-proof decision schemes but strategy proofness is again inconsistent with efficiency of the mechanism. The economic implications of these strategy-proof mechanisms are numerous and have been systematically investigated (on the vast literature about Clarke-Groves mechanisms see, e.g., the special supplement of the Spring 1977 issue of *Public Choice*).

In view of applications to political science, side payments or lotteries over alternatives are hardly justifiable. If we think of decision schemes as representing voting procedures, then we need a relaxation of the unre-

* The author wishes to express thanks to S. Barbera for fruitful discussions and pertinent advice. Thanks are also due an anonymous referee for helpful comments.

Public Choice 35: 437-455 (1980) 0048-5829/80/0354-0437 $2.85.

stricted domain condition which is, in particular, consistent with pure ordinality of the preference profile. Such a relaxation has been proposed many times in the literature: it is the well-known assumption that the preferences of the agents are all single-peaked. As early as 1961, Dummett and Farquharson [4] noticed that the Condorcet winner (which in this context is very simply the median peak) yields a social choice function immune to any strategic manipulation by an agent or group of agents (see also reference [10]).

In this paper we intend to deal with the strategy-proof decision schemes in this third context where the preferences of the agents are all single-peaked along the real line. Assuming that the agents are all aware of this 'restricted domain', most of the pertinent information about a particular preference is described by its 'peak' alternative. Therefore it is natural to consider only those voting schemes where each agent simply announces his peak-alternative (lies being allowed). Within this framework we characterize all strategy-proof voting schemes. It turns out that the Condorcet winner is not the only strategy-proof voting scheme: actually every strategy-proof, efficient and anonymous voting scheme is obtained by adding $(n-1)$ fixed ballots to the n voters' ballots and then choosing the median of this larger set of ballots.[1] This provides a larger class of procedures all resembling the median peak procedure within which the collectivity as a whole influences the final decision without violating the efficiency requirement.

1. Strategy-proof voting schemes in which the preferences are single-peaked

Let us consider a world with n agents and a set A of alternatives. We denote U_i the set of possible preferences of agent i (it is a subset of the set of preference preorderings of A).

Let us say that alternative a 'defeats by majority vote' alternative b if the set of agents that strictly prefer a to b contains at least $(p+1)$ agents (we assume temporarily that $n = 2p+1$ is odd).

Suppose that the domains $U_1, \ldots, U_n$ are such that for every profile $(u_1, \ldots, u_n) \in U_1 \times \ldots \times U_n$ there exists a (necessarily unique) Condorcet winner, that is, an alternative $C(u_1, \ldots, u_n)$ that defeats every other alternative by majority vote. We now claim that the social choice function $(u_1, \ldots, u_n) \to C(u_1, \ldots, u_n)$ is non manipulable by any agent or coalition of agents.

The proof of this result – which is found in the literature as early as 1948 (see [1,2]) – is elementary and very close to the proof of Proposition 1 below.

From now on, we limit ourselves to the traditional context where a Condorcet winner always exists: we assume that $A = \mathcal{R}$ is the real line,[2] and that for every i, U_i is the set S of single-peaked preferences, that is to say:

$$\left\{\begin{array}{ll} \multicolumn{2}{l}{u \in S \text{ if and only if there exists an alternative } a \text{, the 'peak' of } u,} \\ \multicolumn{2}{l}{\text{such that:}} \\ \text{for all } x, y \in \mathcal{R} & \left\{\begin{array}{l} x \leqslant y < a \Rightarrow u(x) \leqslant u(y) < u(a) \\ a < x \leqslant y \Rightarrow u(a) > u(x) \geqslant u(y) \end{array}\right. \end{array}\right. \tag{1}$$

(Notice that we identify the preference preordering u with any utility function associated with it.)

Let $(u_1, \ldots, u_n) \in S^n$ be a profile with the corresponding peaks $(a_1, \ldots, a_n)$. If $n = 2p + 1$ is odd, the Condorcet winner is the 'median' peak denoted by $m(a_1, \ldots, a_n)$ and defined by

$$m(a_1, \ldots, a_n) = m \text{ if and only if } \left\{\text{and } \begin{array}{l} \# \{i/a_i \leqslant m\} \geqslant p + 1 \\ \# \{i/a_i \geqslant m\} \geqslant p + 1 \end{array}\right. \tag{2}$$

(where $\# Z$ denotes the cardinality of Z). Note that $m(a_1, \ldots, a_n)$ is one of the a_i.

As a consequence of the result stated above we obtain that the corresponding social choice functions are strategy-proof and group-strategy-proof. Let us state this property precisely. A basic assumption of the model is that the agents are mutually aware that their preferences are single-peaked: they are not allowed[3] to announce non single-peaked orderings. Throughout the paper we make an additional assumption which implies some loss of generality in our model: we assume that each agent's message is simply his peak. Accordingly we define a voting scheme as a mapping π from $\mathcal{R}^n$ into $\mathcal{R}$ which associates with every n-tuple $(x_1, \ldots, x_n)$ of announced peaks the selected alternative $\pi(x_1, \ldots, x_n)$. Of course a larger class of schemes is obtained by considering all mappings π^* from S^n into $\mathcal{R}$, that is all decision-making mechanisms where each agent's message is to announce an entire single-peaked preference. To every voting scheme π (defined as above) we can obviously associate a mapping π^* defined by:

$$\begin{array}{l} \forall (u_1, \ldots, u_n) \in S^n \\ \forall (a_1, \ldots, a_n) \in \mathcal{R}^n \end{array} \left[\begin{array}{l} \forall i = 1, \ldots, n \\ a_i \text{ is the peak} \\ \text{of } u_i \end{array}\right] \Rightarrow \left[\begin{array}{l} \pi^*(u_1, \ldots, u_n) \\ = \pi(a_1, \ldots, a_n) \end{array}\right]$$

Therefore our results characterizing the strategy-proof voting schemes π provide a great deal of information about strategy-proof decision-making mechanisms π^* (since for every strategy-proof π, the associated mechanism π^* is strategy-proof as well). But we do not give a complete characterization of strategy-proof mechanisms π^*.[4] We can say that the voting scheme π is *strategy proof* if for every agent i, and for every single-peaked preference $u_i \in S$ with associated peak a_i, we have:

$$\forall x_i \in \mathcal{R} \ \forall x_{\hat{i}} \in \mathcal{R}^{n-1} \ u_i(\pi(a_i, x_{\hat{i}})) \geqslant u_i(\pi(x_i, x_{\hat{i}}))$$

(where $x_{\hat{i}}$ is the $(n-1)$-uple of peaks announced by the other agents).

We can say that π is *group-strategy-proof* is for every coalition $S \subset \{1, \ldots, n\}$ for every preference profile $(u_i)_{i \in S} \in S^S$ with associated peaks $a_S = (a_i)_{i \in S}$ we have:

$$\forall x_{S^c} \in \mathcal{R}^{S^c} \ \exists x_S \in \mathcal{R}^S \ \forall i \in S$$

$$u_i(\pi(x_S, x_{S^c})) > u_i(\pi(a_S, x_{S^c})) \tag{3}$$

The concept of strategy-proofness corresponds to the non-cooperative stability of the Nash equilibrium: no agent has an incentive to announce any other alternative than his true peak. In a group-strategy proof voting scheme, this stability property holds for coalitions as well: no coalition of players has an incentive to collectively misrepresent their peaks. Relation (3) above amounts to saying that for every profile $(u_1, \ldots, u_n) \in S^n$ the n-uple $(a_1, \ldots, a_n)$ of associated peaks is a *strong equilibrium* of the normal form game $(\mathcal{R}, \ldots, \mathcal{R}, u_1 \circ \pi, \ldots, u_n \circ \pi)$.

Examples of group-strategy-proof voting schemes are the Condorcet voting schemes:

$$\pi(x_1, \ldots, x_n) = m(x_1, \ldots, x_n) \text{ if } n \text{ is odd} \tag{4}$$

Actually the above voting schemes are particular members of a larger class of group-strategy-proof voting schemes that we will now describe. Let k be an integer such that $1 \leqslant k \leqslant n$. If the real numbers $x_1, \ldots, x_n$ are re-ordered by increasing value, we denote by $\pi_k(x_1, \ldots, x_n)$ the number ranked k-th. These voting schemes, again, are group-strategy-proof. For $k = 1$ and $k = n$ we obtain in particular:

$$\pi_1(x_1, \ldots, x_n) = \inf(x_1, \ldots, x_n) \tag{5}$$

$$\pi_n(x_1, \ldots, x_n) = \sup(x_1, \ldots, x_n) \tag{6}$$

Voting scheme (5) is interpreted as follows: low alternatives are socially praised: only the unanimous coalition can push the selected alternative above any fixed level.

The following proposition describes a class of group-strategy-proof voting schemes including all the schemes previously described.

Proposition 1

Let $\alpha_1, \ldots, \alpha_{n-1}$ be $(n-1)$ real numbers possibly equal to $+\infty$ or $-\infty$:

$$\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{R} \cup \{+\infty, -\infty\}$$

The following voting scheme

$$\pi: \mathbb{R}^n \to \mathbb{R} \quad \pi(x_1, \ldots, x_n) = m(x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_{n-1}) \tag{7}$$

is then group-strategy-proof, anonymous (symmetric with respect to the players), and efficient (the selected alternative is Pareto optimal).

Note that in definition (7) we extend directly the definition of m to $(\mathbb{R} \cup \{\pm\infty\})^{2n-1}$.

Nevertheless the values of π are always finite.

Proof

Anonymity of π is clear.

If the peaks of the agents are $a_1, \ldots, a_n$, then the Pareto set is

$$[\inf_i x_i, \sup_i x_i] \quad .$$

Therefore efficiency of π is a result of the following remark:

$$\text{Fix } (y_1, \ldots, y_{2n-1}) \in \mathbb{R}^{2n-1}$$

then

$$\# \{ i/y_i \geqslant \inf_j y_j, j = 1, \ldots, n\} \geqslant n$$

Therefore

$$m(y_1, \ldots, y_{2n-1}) \geqslant \inf_j y_j \text{, for } j = 1, \ldots, n$$

and similarly

$$\# \{ i/y_i \leqslant \sup_j y_j, j = 1, \ldots, n\} \geqslant n$$

implying

$$m(y_1, \ldots, y_{2n-1}) \leqslant \sup_j y_j \text{, for } j = 1, \ldots, n$$

That π is group-strategy-proof remains to be proved. Suppose the contrary and let $(u_i)_{i=1,\dots,n} \in S^n$ be a preference profile with associated peaks $(a_1, \dots, a_n)$ such that the agents of coalition S have an incentive to announce $x_S \in \mathbb{R}^S$ instead of a_S:

$$\forall\, i \in S \;\; u_i(\pi(x_S a_{S^c})) > u_i(\pi(a_S, a_{S^c})) \tag{8}$$

By (8) we have $\pi(x_S, a_{S^c}) \neq \pi(a_S, a_{S^c})$; suppose:

$$\pi(x_S, a_{S^c}) > \pi(a_S, a_{S^c}) \tag{9}$$

Because each u_i is single peaked (see (1)), inequalities (8) and (9) together imply

$$\forall\;\; i \in S \;\; a_i > \pi(a_S, a_{S^c}) \tag{10}$$

By definition of $\pi(a_S, a_{S^c}) = m(a_1, \dots, a_n, \alpha_1, \dots, \alpha_{n-1})$, if we set $(a_1, \dots, a_n, \alpha_1, \dots, \alpha_{n-1}) = (y_1, \dots, y_{2n-1})$ we have:

$$\#\{j \in \{1, \dots, (2n-1)\}/y_j \leqslant \pi(a_S, a_{S^c})\} \geqslant n$$

Every y_i such that $y_i \leqslant \pi(a_S, a_{S^c})$ corresponds either to some a_i with $i \notin S$ (by (10)) or to some α_i.

Therefore if we set $(x_S, a_{S^c}, \alpha_1, \dots, \alpha_{n-1}) = (z_1, \dots, z_{2n-1})$ we have:

$$\#\{j \in \{1, \dots, 2n-1\}/z_j \leqslant \pi(a_S, a_{S^c})\} \geqslant n \tag{11}$$

Note that for every real number α,

$$n \leqslant \#\{j \in \{1, \dots, 2n-1\}/z_j \leqslant \alpha\} \Rightarrow m(z_1, \dots, z_{2n-1}) \leqslant \alpha$$

Then (11) implies:

$$\pi(x_S, x_{S^c}) = m(z_1, \dots, z_{2n-1}) \leqslant \pi(a_S, a_{S^c})$$

This is the desired contradiction. Q.E.D.

We shall now comment on the family of voting schemes introduced in Definition 1.

If none of the numbers $\alpha_1, \dots, \alpha_{n-1}$ is finite, then the corresponding voting scheme π is one of the π_k described above or more precisely:

$$m(x_1, \dots, x_n, -\infty, \dots, -\infty) = \inf\{x_1, \dots, x_n\}$$
$$m(x_1, \dots, x_n, +\infty, -\infty, \dots, -\infty) = \pi_2(x_1, \dots, x_n)$$

$$m(x_1, \ldots, x_n, +\infty, +\infty, -\infty, \ldots, -\infty) = \pi_3(x_1, \ldots, x_n)$$

$$\vdots$$

$$m(x_1, \ldots, x_n, +\infty, \ldots, +\infty) = \sup\{x_1, \ldots, x_n\}$$

For $n/2 \leqslant k \leqslant n$, we say that in voting scheme π_k, high alternatives are socially praised and a quota of k agents is required to force the selected alternative below any fixed level.

Let us now consider the case of a voting scheme (7) where $\alpha_1, \ldots, \alpha_{n-1}$ are all finite: this amounts to saying that 'society' has $(n-1)$ votes whereas each individual has one single vote; therefore unanimous agents can enforce any arbitrary alternative, but as soon as the agent's preferences differ, then the 'social votes' $\alpha_1, \ldots, \alpha_{n-1}$ arbitrate among them (for instance if every x_i is greater than sup α_j, then the selected alternative inf x_i is the closest alternative to $\{\alpha_1, \ldots, \alpha_{n-1}\}$).[5]

We can note finally that voting schemes (7) contain also the schemes where 'society' has only $(n-3)$ votes (take $\alpha_1 = +\infty$, $\alpha_2 = -\infty$, next $\alpha_3, \ldots, \alpha_n$ finite) or $(n-5)$ votes (take $\alpha_1 = \alpha_2 = +\infty$, $\alpha_3 = \alpha_4 = -\infty$, next $\alpha_5, \ldots, \alpha_n$ finite), and so on.

2. The characterization theorem

Theorem

The following two statements are equivalent:

(i) the voting scheme π from $\mathbb{R}^n$ into $\mathbb{R}$ is strategy-proof, anonymous, and efficient;

(ii) there exist $(n-1)$ real numbers $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{R} \cup \{+\infty, -\infty\}$ such that:

$$\forall (x_1, \ldots, x_n) \in \mathbb{R}^n \quad \pi(x_1, \ldots, x_n) = m(x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_{n-1}) \tag{12}$$

Notice that for anonymous and efficient voting schemes, strategy proofness is equivalent to group-strategy-proofness (this follows from Proposition 1 and the theorem, as well as from Proposition 2 below).

Proposition 2

The voting scheme π from $\mathbb{R}^n$ into $\mathbb{R}$ is strategy-proof and anonymous if and only if there exist $(n+1)$ real numbers $\alpha_1, \ldots, \alpha_{n+1} \in \mathbb{R} \cup \{+\infty, -\infty\}$ such that:

$$\forall \ (x_1, \ldots, x_n) \in \mathbb{R}^n \quad \pi(x_1, \ldots, x_n) = m(x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_{n+1})$$

In particular a strategy-proof anonymous voting scheme is group-strategy-proof.

Proof of the theorem and Proposition 2 is found in the Appendix.

Note that Σ_n, the set of strategy-proof voting schemes, is stable by the operation of supremum and infimum (not necessarily finite). Actually one proves easily that Σ_n is the smallest subset of $\mathcal{R}^{\mathcal{R}^n}$ stable by supremum and infimum and containing the elementary functions:

$$\begin{cases} \pi^i(x_1, \ldots, x_n) = x_i & \text{for } i = 1, \ldots, n \quad \text{for } a \in \mathcal{R} \\ {}_a\pi(x_1, \ldots, x_n) = a & \end{cases}$$

In particular, the sequence $\Sigma_1, \Sigma_2, \ldots, \Sigma_n, \ldots$ has the following decentralization property: suppose that the set of n agents is partitioned into p-coalitions with respective cardinality $n_1, \ldots, n_p$. Let $\pi_1 \in \Sigma_{n_1}, \ldots, \pi_p \in \Sigma_{n_p}$ be strategy-proof voting schemes among the agent coalitions of the partition. Finally, let $\pi_0 \in \Sigma_p$ be a strategy-proof voting scheme among p players. Then the compound voting scheme π:

$$\pi(x_1, \ldots, x_n) = \pi_0(\pi_1(x_1, \ldots, x_{n_1}), \pi_2(x_{n_1+1}, \ldots, x_{n_1+n_2}), \ldots)$$

belongs to Σ_n. This amounts to saying that if the final decision is taken by a two-stage procedure where the initial committee is partitioned into subcommittees, each of which has to select a 'representative', then no manipulation may arise in the procedure as a whole if each stage is independently non manipulable. In other words, representatives need not be strategically mandated.

As a final result we can observe that a strategy-proof voting scheme is also group-strategy-proof (the proof is similar to the proof of Proposition 1 and left to the reader).

Conclusion

This paper investigates one of the possible weakening of the (too demanding) assumptions of the Gibbard-Satterthwaite theorem. Namely we deal with a class of voting schemes where at the same time the domain of possible preference preordering of any agent is limited to single-peaked preferences, and the message that this agent sends to the central authority is simply its 'peak' – his best preferred alternative. In this context we have shown that strategic considerations justify the central role given to the Condorcet procedure which amounts to elect the 'median' peak: namely all strategy-proof anonymous and efficient voting schemes can be derived from the Condorcet procedure by simply adding some fixed ballots to the agent's ballots (with the only restriction that the number of fixed ballots is strictly less than the number of agents).

Therefore, as long as the alternatives can be ordered along the real line with the preferences of the agents being single-peaked, it makes little sense to object against the Condorcet procedure, or one of its variants that we display in our characterization theorem.

An obvious topic for further research would be to investigate reasonable restrictions of the domain of admissible preferences such that a characterization of strategy-proof voting schemes can be found. The single-peaked context is obviously the simplest one, allowing very complete characterizations. When we go on on to the two-dimensional state of alternatives the concept of single peakedness itself is not directly extended and a generalization of our one-dimensional results seems to us to be a difficult but motivating goal.

NOTES

1. Adding fixed ballots to the voters' ballot is a technical device already used by Murakami to describe the so-called representative systems of social functions between only two alternatives (see [5, 9]). This can be viewed as a special case of our very formalism (see note 5).
2. Actually all the results of this paper are easily transposable to the somewhat more elementary context where A is finite and its elements are linearly ordered arbitrarily.
3. Blin and Satterthwaite prove in [3] that allowing the players to announce any ordering would remove the strategy proofness of the voting scheme.
4. Actually one checks easily that some strategy-proof mechanisms π^* do not derive from one of the strategy-proof voting schemes π described in the results stated below. Consider for instance the following mechanism involving a single agent ($n = 1$):

   For every $z \in S$ with associated peak a, we take:

   $$\begin{array}{llll} \pi^*(z) = a & \text{if } a \leqslant -1 & \text{or} & a \geqslant +1 \\ \pi^*(z) = -1 & \text{if } -1 \leqslant a \leqslant 1 & \text{and} & z(-1) > z(1) \\ \pi^*(z) = +1 & \text{if } -1 \leqslant a \leqslant 1 & \text{and} & z(-1) \leqslant z(1) \end{array}$$

   Clearly this mechanism is strategy-proof (it does not pay to announce a false z) but it cannot be derived from a voting scheme in the above sense.
5. If the agents have only to decide among two alternatives a, b then we obtain Murakami's elementary voting schemes of the form:

   $m(x_1, \ldots, x_n, \underbrace{a, \ldots, a}_{k \text{ times}})$ where agent i's ballot x_i is a or b

   $m(x_1, \ldots, x_n, \underbrace{b, \ldots, b}_{k \text{ times}})$ where agent i's ballot x_i is a or b.

REFERENCES

[1] Black, D. (1948). 'The Decision of a Committee Using a Special Majority.' *Econometrica*, 16.
[2] Black, D. (1948). 'On the Rationale of Group Decision-Making.' *Journal of Political Economy*, 56.
[3] Blin, J.M., and Satterthwaite, M.A. (1976). 'Strategy-Proofness and Single Peakedness.' *Public Choice*, 26: 51.
[4] Dummett and Farquharson. (1961). 'Stability in Voting.' *Econometrica*, 29: 33-44.
[5] Fishburn, P. (1973). *The Theory of Social Choice*. Princeton: Princeton University Press.
[6] Gibbard, A. (1973). 'Manipulation of Voting Schemes: A General Result.' *Econometrica*, 41: 587-601.
[7] Gibbard, A. (1977). 'Manipulation of Schemes That Mix Voting with Chance.' *Econometrica*, 45 (3).
[8] Gibbard, A. (1978). 'Straightforwardness of Game Forms with Lotteries as Outcomes.' *Econometrica*, 46 (3): 595.
[9] Murakami, Y. (1968). *Logic and Social Choice*. London: Routledge and Kegan Paul.
[10] Pattanaik. (1974). *Collective Rationality of Group-Decision Rules*. Working Paper 138. School of Economics, University of Delhi.

APPENDIX

Proof of the theorem and Proposition 2

Step 1
For every integer n, $n \geqslant 1$, we denote by S_n the following subset of $\mathcal{R}^{\mathcal{R}^n}$:

$$S_n = \{\pi \colon \mathcal{R}^n \to \mathcal{R} \mid \exists\ \alpha_1, \ldots, \alpha_{n+1} \in \mathcal{R} \cup \{+\infty, -\infty\}:$$

$$\pi(x_1, \ldots, x_n) = m(x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_{n+1})\}$$

Every element of S_n is clearly a group-strategy-proof and anonymous voting scheme: the proof follows word for word the proof of Proposition 1. Only efficiency is violated since we have for instance:

$$m(x_1, \ldots, x_n, \underbrace{\alpha, \ldots, \alpha}_{(n+1)\text{ times}}) = \alpha$$

We will now prove that, conversely, every strategy-proof anonymous voting scheme belongs to S_n. This will prove Proposition 2 and in turn imply the theorem, namely, if π belongs to S_n and moreover, is efficient, we have:

$$\forall\ x \in \mathcal{R}\ m(\underbrace{x, \ldots, x}_{n\text{ times}}, \alpha_1, \ldots, \ldots \alpha_{n+1}) = \pi(x, \ldots, x) = x \qquad (13)$$

Thus $\alpha_i > -\infty$ for every $i = 1, \ldots, (n+1)$ is impossible (it would contradict (13) for $x < \inf \alpha_i$) as well as $\alpha_i < +\infty$ for every $i = 1, \ldots, (n+1)$ (it would contradict (13) for $x > \sup \alpha_i$). Therefore at least one of the α_i's is $+\infty$ and one is $-\infty$.

Since $m(+\infty, -\infty, \beta_1, \ldots, \beta_{2k+1}) = m(\beta_1, \ldots, \beta_{2k+1})$ we can drop two of the α_i's in $m(x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_{n+1})$. This proves that π has the desired form (7).

Step 2

We now prove by induction on n that every strategy-proof anonymous voting scheme among n agents belongs to S_n. We start by the rather trivial 1-agent voting schemes. To say that π from $\mathcal{R}$ into $\mathcal{R}$ is strategy-proof is to say:

$$\forall\ x \in \mathcal{R}\quad \forall\ u \in S \text{ (the peak of } u \text{ is } a) \Rightarrow u(\pi(a)) \geqslant u(\pi(x))$$

Choose two numbers x, $a \in \mathcal{R}$ and suppose:

$$\pi(a) < \inf\{a, \pi(x)\}$$

If we have $\pi(a) < \pi(x) \leqslant a$ then the single-peaked utility function $u(y) = -|y - a|$, with peak a, is such that:

$$u(\pi(a)) < u(\pi(x))$$

If we have $\pi(a) < a < \pi(x)$ then the following single-peaked utility function with peak a:

$$u(y) = \begin{cases} \dfrac{\pi(x) - \pi(a)}{a - \pi(a)} \cdot (y - a) & \text{if } y \leqslant a \\ a - y & \text{if } y \geqslant a \end{cases}$$

is such that:

$$u(\pi(a)) = \pi(a) - \pi(x) < a - \pi(x) = u(\pi(x))$$

We then obtain that for every $x, a \in \mathbb{R}$

$$\pi(a) \geqslant \inf\{a, \pi(x)\}$$

A symmetrical argument proves that

$$\pi(a) \leqslant \sup\{a, \pi(x)\}$$

Thus the function π is such that:

$$\forall\ x, y \in \mathbb{R} \quad \pi(x) \in [x, \pi(y)] \tag{14}$$

Let us set:

$$\begin{cases} \alpha = \inf\limits_{x\in\mathbb{R}} \pi(x) \in \mathbb{R} \cup \{-\infty\} \\ \beta = \sup\limits_{x\in\mathbb{R}} \pi(x) \in \mathbb{R} \cup \{+\infty\} \end{cases}$$

For every $x \in \mathbb{R}$, we deduce from (14) that $\pi(x) \in [x, \alpha]$ and therefore for $x \leqslant \alpha$ (if α is finite) we have:

$$\alpha \leqslant \pi(x) \in [x, \alpha] \Rightarrow \pi(x) = \alpha$$

A symmetrical argument yields

$$\forall\ x \in \mathbb{R}:\ \pi(x) \in [x, \beta]$$

Therefore for $x \geqslant \beta$ (if β is finite) we obtain $\pi(x) = \beta$.

For every x such that $\alpha \leqslant x \leqslant \beta$ we now have

$$\pi(x) \in [x, \alpha] \cap [x, \beta] = \{x\}$$

Our function π is as follows

$$\begin{aligned} \pi(x) &= \alpha && \text{if } x \leqslant \alpha \\ &= x && \text{if } \alpha \leqslant x \leqslant \beta \\ &= \beta && \text{if } x \geqslant \beta \end{aligned}$$

Thus $\pi(x) = m(x, \alpha, \beta)$ for all x: this proves that π belongs to S_1.

Step 3

We now suppose that our claim holds true for n and we prove it for $(n + 1)$.

Let $\pi(x_0, x_1, \ldots, x_n)$ be an anonymous strategy-proof voting scheme among $(m+1)$ players. If we fix x_0 then $(x_1, \ldots, x_n) \to \pi(x_0, x_1, \ldots, x_n)$ is clearly an anonymous strategy-proof voting scheme among n players. By the induction assumption, it belongs to S_n. There then exist $(n + 1)$ functions $\alpha_1, \ldots, \alpha_{n+1}$ from $\mathbb{R}$ into $\mathbb{R} \cup \{+\infty, -\infty\}$ such that:

$$\forall\ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$$
$$\pi(x_0, x_1, \ldots, x_n) = m(x_1, \ldots, x_n, \alpha_1(x_0), \ldots, \alpha_{n+1}(x_0))$$

Up to a possible redefinition of the α_i's we can assume:

$$\forall\ x_0 \in \mathbb{R}\ \ \alpha_1(x_0) \leqslant \ldots \leqslant \alpha_{n+1}(x_0) \tag{15}$$

If we now fix $(x_1, \ldots, x_n)$ then $x_0 \to \pi(x_0, x_1, \ldots, x_n)$ belongs to S_1 and therefore verifies property (14):

$$\forall\ x_0, x_0' \in \mathbb{R}\ \ \forall\, (x_1, \ldots, x_n) \in \mathbb{R}^n$$
$$\pi(x_0, x_1, \ldots, x_n) \in [x_0, \pi(x_0', x_1, \ldots, x_n)] \tag{16}$$

We now fix $x_0 \in \mathbb{R}$ and an index k, $1 \leqslant k \leqslant (n + 1)$.

We can remark that:

$$\lim_{\lambda \to +\infty} m(\underbrace{\lambda, \ldots, \lambda}_{(k-1)\text{ times}}, \underbrace{-\lambda, \ldots, -\lambda}_{(n-k+1)\text{ times}}, \alpha_1(x_0), \ldots, \alpha_{n+1}(x_0)) = \alpha_k(x_0)$$

(the proof of this claim is elementary and left to the reader).

Applying (16) for fixed x_0, x_0' and with:

$$(x_1, \ldots, x_n) = (\underbrace{\lambda, \ldots, \lambda}_{(k-1)}, \underbrace{-\lambda, \ldots, -\lambda}_{(n-k-1)})$$

we obtain for every index k:

$$\forall\ x_0, x_0' \in \mathbb{R}\ \ \alpha_k(x_0) \in [x_0, \alpha_k(x_0')]$$

This property implies that α_k belongs to S_1 (see Step 2: that α_k takes some infinite values does not affect the argument).

Then α_k can be written as:

$$\alpha_k(x_0) = m(x_0, a_k, b_k) \text{ where } -\infty \leqslant a_k \leqslant b_k \leqslant +\infty$$

and our voting scheme π is written as:

$$\pi(x_0, x_1, \ldots, x_n) = m(x_1, \ldots, x_n, m(x_0, a_1, b_1), \ldots, m(x_0, a_{n+1}, b_{n+1}))$$

Using anonymity of π, we now prove:

$$b_1 = a_2, \ldots, b_k = a_{k+1}, \ldots, b_n = a_{n+1} \tag{17}$$

First of all we have by (15) and for all k, $1 \leqslant k \leqslant n$:

$$\forall\ x_0 \in \mathbb{R} \quad m(x_0, a_k, b_k) \leqslant m(x_0, a_{k+1}, b_{k+1})$$

which is equivalent to $a_k \leqslant a_{k+1}$ and $b_k \leqslant b_{k+1}$.

Suppose $a_{k+1} < b_k$: we can then choose $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ such that:

$$\begin{cases} a_{k+1} \leqslant x_0 < x_n \leqslant b_k \\ x_1 = \ldots = x_{n-k} = \lambda < x_0 \\ x_{n-k+1} = \ldots = x_{n-1} = \mu > x_n \end{cases}$$

This implies:

$$\forall\ k' \leqslant k-1:$$

$$\alpha_{k'}(x_0) \leqslant \alpha_k(x_0) = \alpha_{k+1}(x_0) = x_0 \leqslant \alpha_{k''}(x_0)$$

for all $k'' \geqslant k+2$

Therefore

$$\pi(x_0, x_1, \ldots, x_n)$$

$$= m(\underbrace{\ldots\lambda\ldots}_{(n-k)}, \underbrace{\ldots\mu\ldots}_{(k-1)}, x_n, \underbrace{\ldots\alpha_k(x_0)\ldots}_{(k-1)}, x_0, x_0, \underbrace{\ldots\alpha_{k''}(x_0)}_{(n-k)\text{ times}} \ldots) = x_0$$

Similarly, we have

$$\forall\ k' \leqslant k-1$$

$$\alpha_{k'}(x_n) \leqslant \alpha_k(x_n) = \alpha_{k+1}(x_n) = x_n \leqslant \alpha_{k''}(x_n)$$

for all $k'' \geqslant k+1$

Therefore:

$$\begin{aligned}
&\pi(x_n, x_1, \ldots, x_{n-1}, x_0) \\
&= m(\underbrace{..\lambda..}_{(n-k)}, \underbrace{..\mu..}_{(k-1)}, x_0, \underbrace{..\alpha_{k'}(x_n)..}_{(k-1)}, x_n, x_n, \underbrace{..\alpha_{k''}(x_n)..}_{(n-k)\text{ times}}) \\
&= x_n
\end{aligned}$$

This assumption $x_0 \neq x_n$ thus contradicts the anonymity of π. We have proved $b_k \leqslant a_{k+1}$.

Suppose now $b_k < a_{k+1}$ we can then choose $(x_0, x_1, \ldots, x_n) \in \mathcal{R}^{n+1}$ such that:

$$\begin{cases} b_k \leqslant x_0 < x_n \leqslant a_{k+1} \\ x_1 = \ldots = x_{n-k} = \lambda < x_0 \\ x_{n-k+1} = \ldots = x_{n-1} = \mu > x_n \end{cases}$$

This implies:

$$\forall\ k' \leqslant k-1$$

$$\alpha_{k'}(x_0) \leqslant \alpha_k(x_0) = b_k < a_{k+1} = \alpha_{k+1}(x_0) \leqslant \alpha_{k''}(x_0)$$

for all $k'' \geqslant k+2$ (18)

Therefore

$$\begin{aligned}
&\pi(x_0, x_1, \ldots, x_n) \\
&= m(\underbrace{..\lambda..}_{(n-k)}, \underbrace{..\mu..}_{(k-1)}, x_n, \underbrace{..\alpha_{k'}(x_0)..}_{(k-1)}, b_k, a_{k+1}, \underbrace{..\alpha_{k''}(x_0)..}_{(n-k)\text{ times}}) \\
&= x_n
\end{aligned}$$

Thus we have proved $b_k = a_{k+1}$, that is (17).

If we set $b_{n+1} = a_{n+2}$ we obtain the following expression of π

$$\pi(x_0, \ldots, x_n) = m(x_1, \ldots, x_n, m(x_0, a_1, a_2), \ldots, m(x_0, a_k, a_{k+1}), \ldots \\ \ldots, m(x_0, a_{n+1}, a_{n+2})) \tag{19}$$

with

$$-\infty \leqslant a_1 \leqslant a_2 \leqslant \ldots \leqslant a_k \leqslant a_{k+1} \leqslant \ldots \leqslant a_{n+2} \leqslant \infty$$

The last step in the proof of the theorem is to establish that for any increasing sequence $a_1, \ldots, a_{n+2}$ of this type, we have for every $x_0, x_1, \ldots, x_n$:

$$m(x_1, \ldots, x_n \underbrace{\ldots m(x_0, a_k, a_{k+1}) \ldots)}_{(n+1) \text{ terms}}$$

$$= \quad m(x_0, x_1, \ldots, x_n, a_1, \ldots, a_{n+2}) \tag{20}$$

Step 4

We prove formula (20).

First suppose $x_0 \leqslant a_1$. The left-hand term in (20) is then:

$$m(x_1, \ldots, x_n, a_1, \ldots, a_{n+1}) = \theta \tag{21}$$

Because $(n + 1)$ agents are a majority we have:

$$a_1 \leqslant \theta \leqslant a_{n+1}$$

Therefore we have $x_0 \leqslant \theta \leqslant a_{n+2}$. We then use the following observation: if $m(y_1, \ldots, y_p) = \theta$ and $y_{p+1} \leqslant \theta \leqslant y_{p+2}$, then $m(y_1, \ldots, y_{p+2}) = \theta$. This implies here:

$$m(x_0, x_1, \ldots, x_n, a_1, \ldots, a_{n+2}) = \theta$$

The proof of formula (20) in the case $x_0 \geqslant a_{n+2}$ is similar.

Suppose now that for some k, $1 \leqslant k \leqslant n + 1$

$$a_k \leqslant x_0 \leqslant a_{k+1}$$

The left-hand term in (20) is then:

$$m(x_1, \ldots, x_n, a_2, \ldots, a_k, x_0, a_{k+1}, \ldots, a_{n+1}) = \theta'$$

Since $a_2 \leqslant \theta' \leqslant a_{n+1}$ we obtain $a_1 \leqslant \theta' \leqslant a_{n+2}$ and by the same observation:

$$m(x_0, x_1, \ldots, x_n, a_1, a_2, \ldots, a_{n+2}) = \theta'$$

This concludes the proof of Proposition 2 and the theorem. Q.E.D.

Although non anonymous voting schemes are much less interesting for the social choice theory, it seems worthwhile to characterize every strategy proof voting scheme.

Proposition 3

The voting scheme π among n agents is strategy-proof if and only if there exists for every subset S of $\{1, \ldots, n\}$ (including the empty set) a real number $a_S \in \mathcal{R} \cup \{\pm \infty\}$ such that:

$$\forall \; (x_1, \ldots, x_n) \in \mathcal{R}^n$$

$$\pi(x_1, \ldots, x_n) = \inf_{S \subset \{1,\ldots,n\}} [\sup_{i \in S} \{x_i, a_S\}] \tag{22}$$

For instance 2-agent strategy-proof voting schemes take the form:

$$\pi(x_1, x_2) = \inf\{a, \sup(x_1, b_1), \sup(x_2, b_2), \sup(x_1, x_2, c)\}$$

In order to describe this voting scheme, let us suppose

$$c < b_i < a \quad \text{for } i = 1, 2$$

(notice that $a \leqslant b_i \Rightarrow a \leqslant \sup(x_i, b_i)$ so that a can be simply removed from the expression of π, and similarly $b_i < c \Rightarrow \sup(x_i, b_i) \leqslant \sup(x_1, x_2, c)$ so that $\sup(x_i, b_i)$ can be removed from the same expression; thus these inequalities hold true if and only if the four terms are relevant in the expression of π). Furthermore, up to a reordering of the agents, we can assume $b_1 \geqslant b_2$. We have for every x_1, x_2:

$$c \leqslant \pi(x_1, x_2) \leqslant a \quad \text{and} \quad \begin{cases} \pi(x_1, x_2) = a \iff x_1, x_2 \geqslant a \\ \pi(x_1, x_2) = c \iff x_1, x_2 \leqslant c \end{cases}$$

Thus the interval $[c, a]$ is imposed for the selected alternative.

If for some i, $x_i < a$ and some j, $x_j > c$ then π is written as:

$$\pi(x_1, x_2) = \inf\{\sup(x_1, b_1), \sup(x_2, b_2), \sup(x_1, x_2)\}$$

If $b_1 = b_2$ this expression is simply $m(x_1, x_2, b)$, a familiar procedure. Otherwise the two agents have different influences on π. We will let the reader illustrate the above expression for the various relative positions of x_1, x_2 and b_1, b_2.

Proof of Proposition 3

Note that in formula (22), if two coalitions S and T are such that $T \subset S$ and $a_T \leqslant a_S$, then for all x,

$$\sup_{i \in T} \{x_i, a_T\} \leqslant \sup_{i \in S} \{x_i, a_S\}$$

so that the right-hand term plays no role in (22) and a_S can be equivalently replaced by a_T.

We can therefore assume:

$$\forall\ T, S : T \subset S \Rightarrow a_S \leqslant a_T \tag{23}$$

Let us denote by Σ_n the set of voting schemes π taking the form (22) for some family $(a_S)_{S \subset \{1,\ldots,n\}}$ of parameters verifying (23). We let the reader check that $S_1 = \Sigma_1$.

We prove Proposition 3 by introduction of n.

Assuming that it holds true until n, we choose a strategy-proof voting scheme $\pi(x_0, \ldots, x_n)$ among $(n + 1)$ players. For every fixed x_0, π is an n-agent strategy-proof voting scheme, thus belonging to Σ_n by the induction assumption. Therefore π can be written as:

$$\pi(x_0, x_1, \ldots, x_n) = \inf_{S \subset \{1,\ldots,n\}} [\sup_{i \in S} \{x_i, a_S(x_0)\}] \tag{24}$$

Let us establish a non empty coalition S_0 and two fixed numbers x_0 and x_0'. If we choose $x_1, \ldots, x_n$ such that:

$$\begin{cases} \forall\ i \in S_0 & x_i = \mu \\ \forall\ i \notin S_0 & x_i = \lambda \end{cases}$$

We then obtain by (24):

$$\lim_{\substack{\lambda \to +\infty \\ \mu \to -\infty}} \pi(x_0, x_1, \ldots, x_n) = \inf_{S \subset S_0} a_S(x_0)$$

By (23) the right-hand term is simply $a_S(x_0)$.

Thus we obtain:

$$\lim_{\substack{\lambda \to +\infty \\ \mu \to -\infty}} \pi(x_0, x_1, \ldots, x_n) = a_{S_0}(x_0)$$

and similarly:

$$\lim_{\substack{\lambda \to +\infty \\ \mu \to -\infty}} \pi(x_0', x_1, \ldots, x_n) = a_{S_0}(x_0')$$

Because for fixed $x_1, \ldots, x_n$, π is an element of S_1, it must verify (14). The above two conditions imply that a_{S_0} also verifies (14) for every non empty S_0. Hence a_{S_0} belongs to $S_1 = \Sigma_1$ and can be written:

$$a_{S_0}(x_0) = \inf \{\alpha_{S_0}, \sup (\beta_{S_0}, x_0)\}$$

A tedious but straightforward computation is now required to check that a function written in the form

$$\inf_{S \subset \{1,\ldots,n\}} \sup_{i \in S} \{x_i, \inf\{\alpha_S, \sup(\beta_S, x_0)\}\}$$

actually takes the form (22).

The last step in the proof of Proposition 3 is to prove that every voting scheme of the type (22) is strategy-proof: one simply verifies that every function (22) satisfies

$$\forall\ i\ \forall\ x_i, x_i' \in \mathbb{R}\ \forall\ x_{\hat{\imath}} \in \mathbb{R}^{n-1}$$

$$\pi(x_i, x_{\hat{\imath}}) \in [x_i, \pi(x_i', x_{\hat{\imath}})] \qquad \text{Q.E.D.}$$