PPP-completeness with Connections to Cryptography

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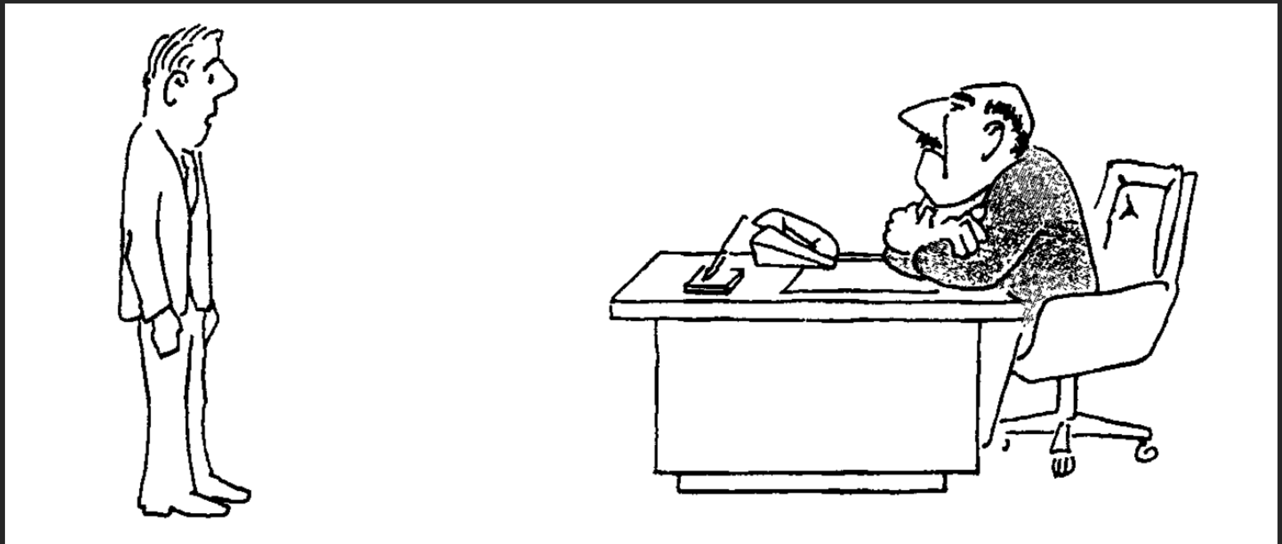
CoReLab Seminar 2020

Motivation via Cryptography

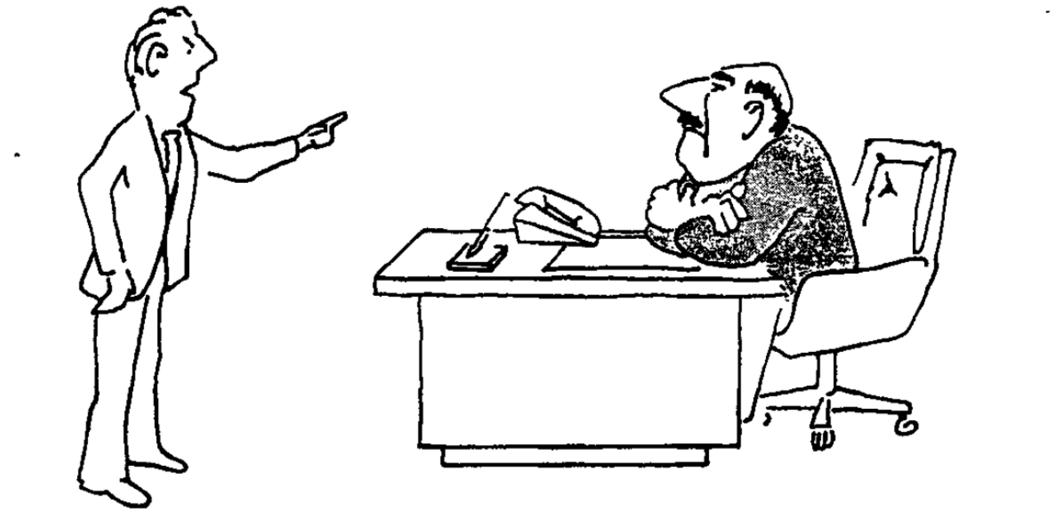
NP-hardness



"I can't find an efficient algorithm, I guess I'm just too dumb."

pictures from "Computers and Intractability" by Garey and Johnson 1979.

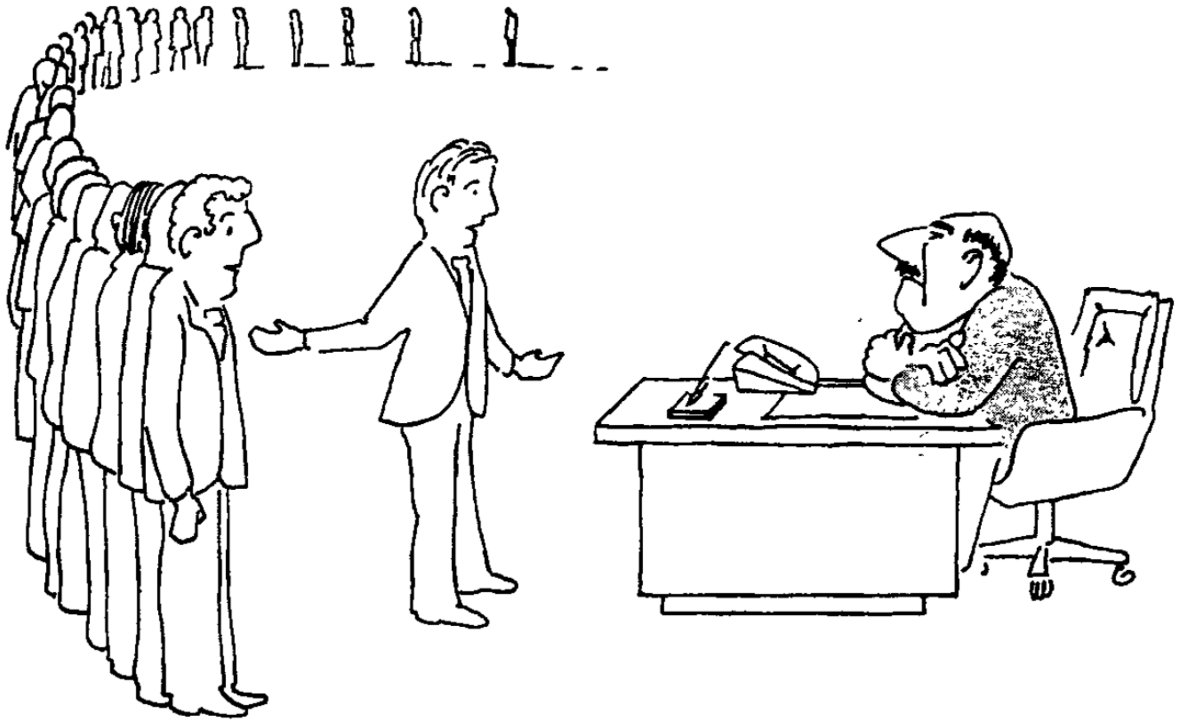
NP-hardness



"I can't find an efficient algorithm, because no such algorithm is possible!"

pictures from "Computers and Intractability" by Garey and Johnson 1979.

NP-hardness



“I can't find an efficient algorithm, but neither can all these famous people.”

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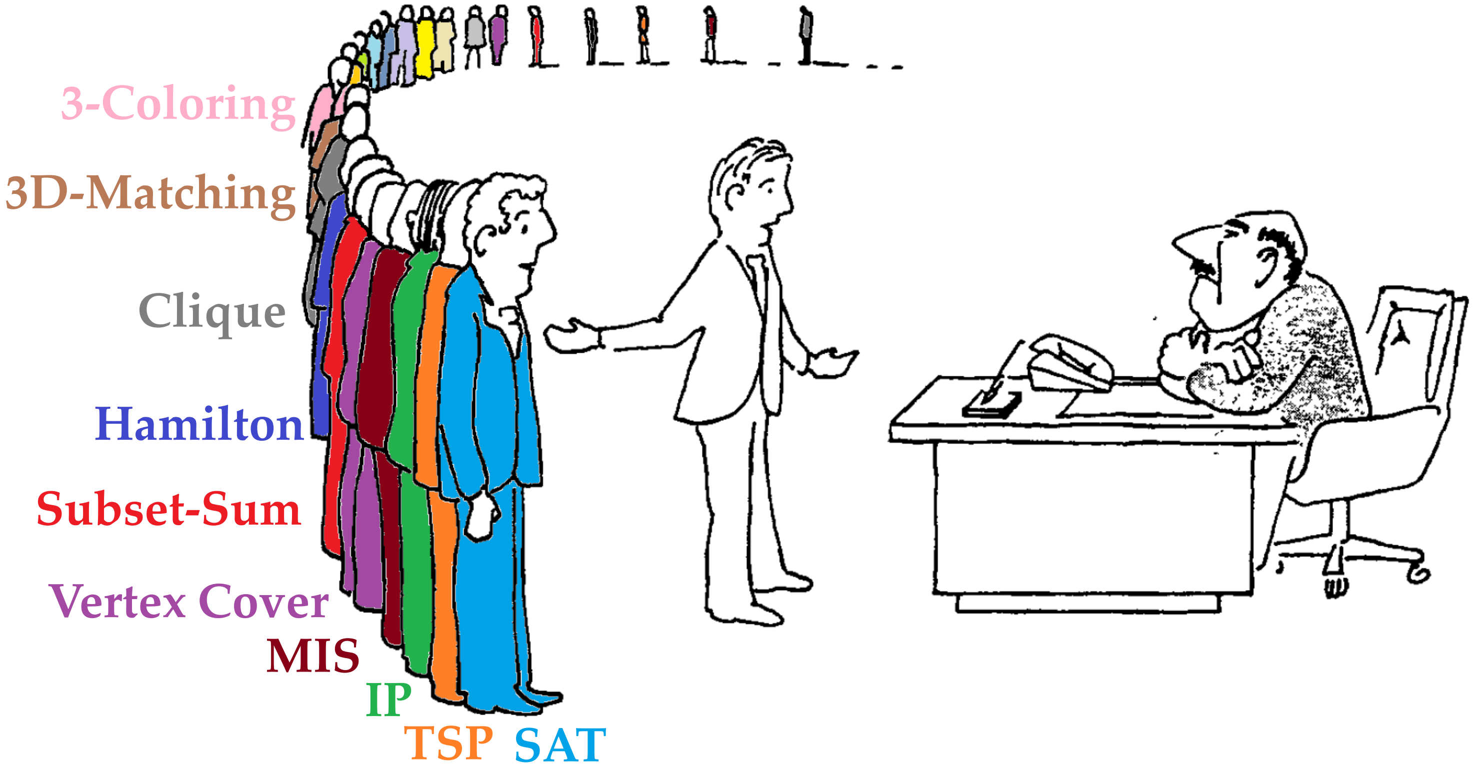
NP-hardness



"If I could find an algorithm I could solve all these famous difficult problems"

Cryptographic Hardness?

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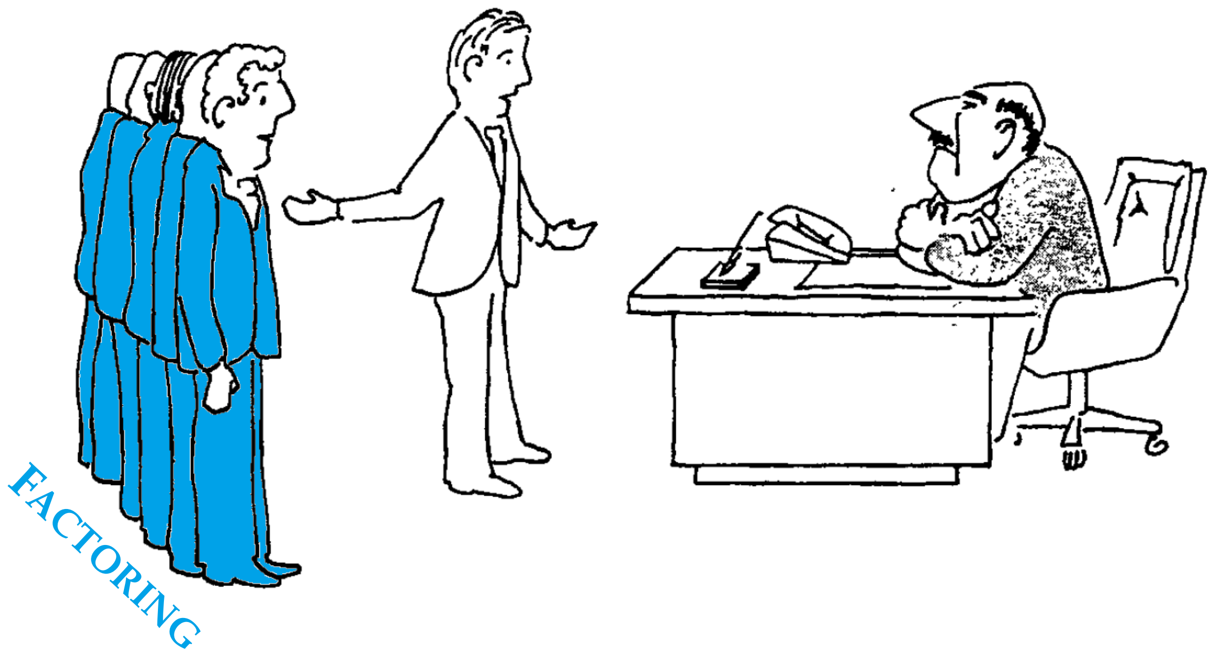
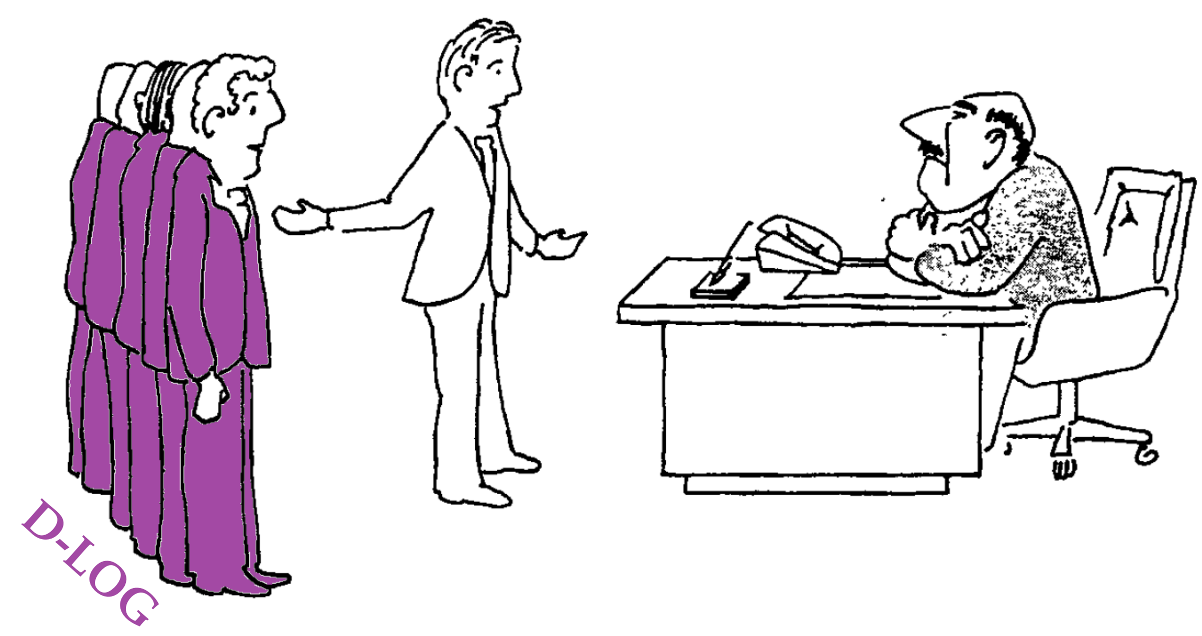


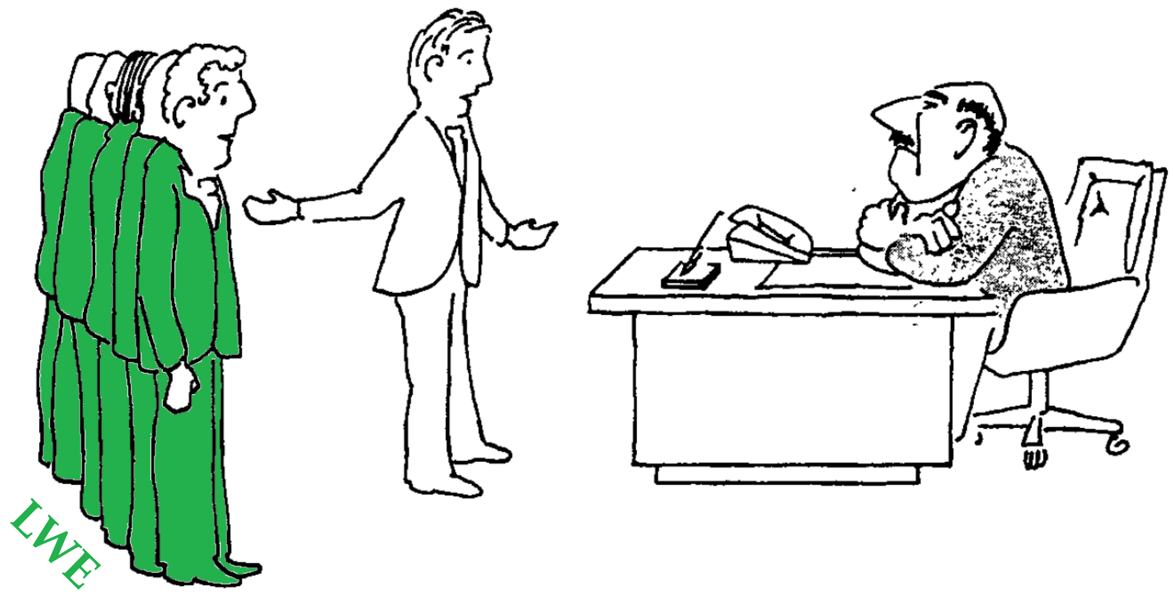
"If someone could break the protocol, they could solve **FACTORING** *on average*."

Cryptographic Hardness



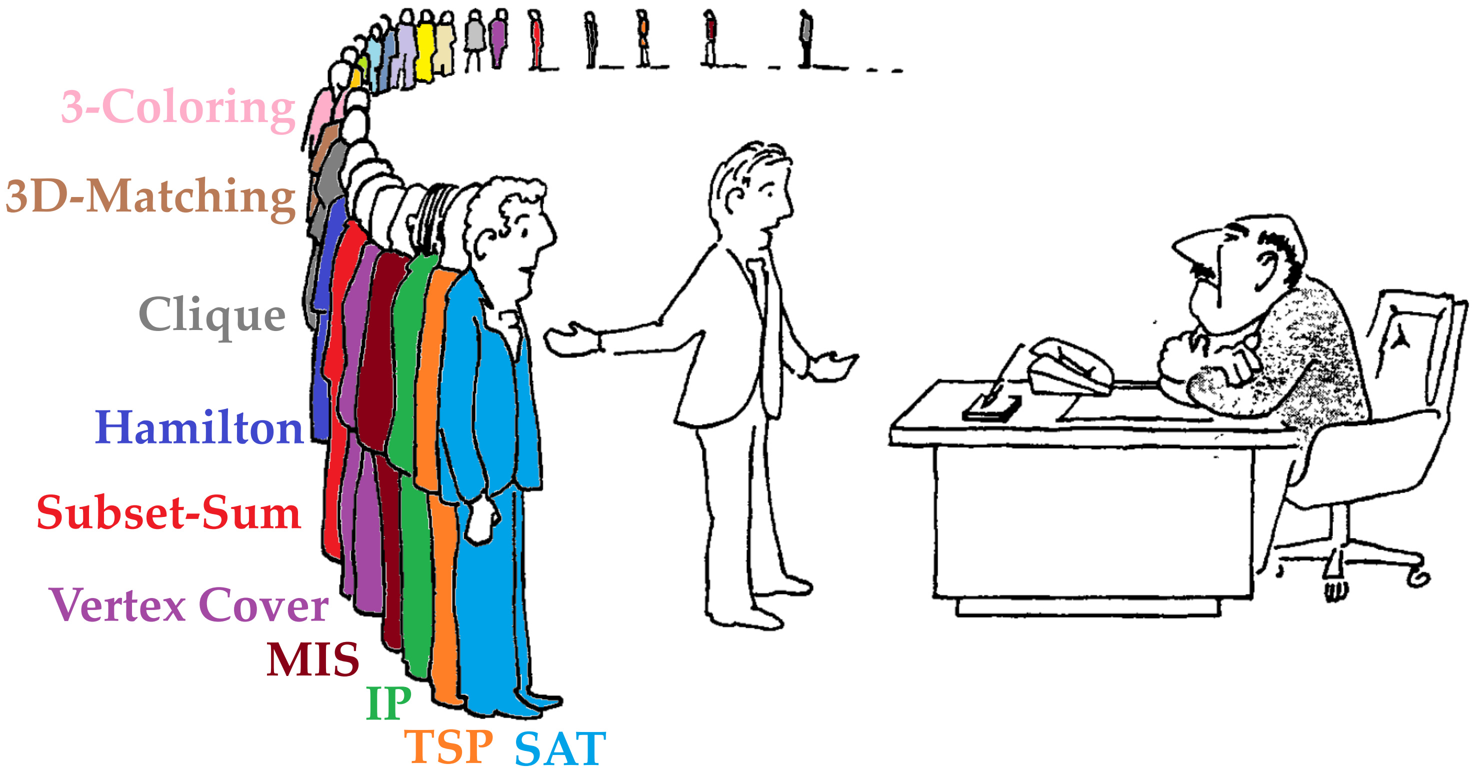
"If someone could break the protocol, they could solve **DISCRETE-LOG** *on average*."

Cryptographic Hardness



"If someone could break the protocol, they could solve **LWE** *on average*."

Utopia Cryptographic Hardness



"If someone could break the protocol, they could solve *on worst-case* all these famous difficult problems"

Can we achieve Cryptographic Utopia?

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Bottlenecks

- cryptography is based on problems that are hard **on average!**

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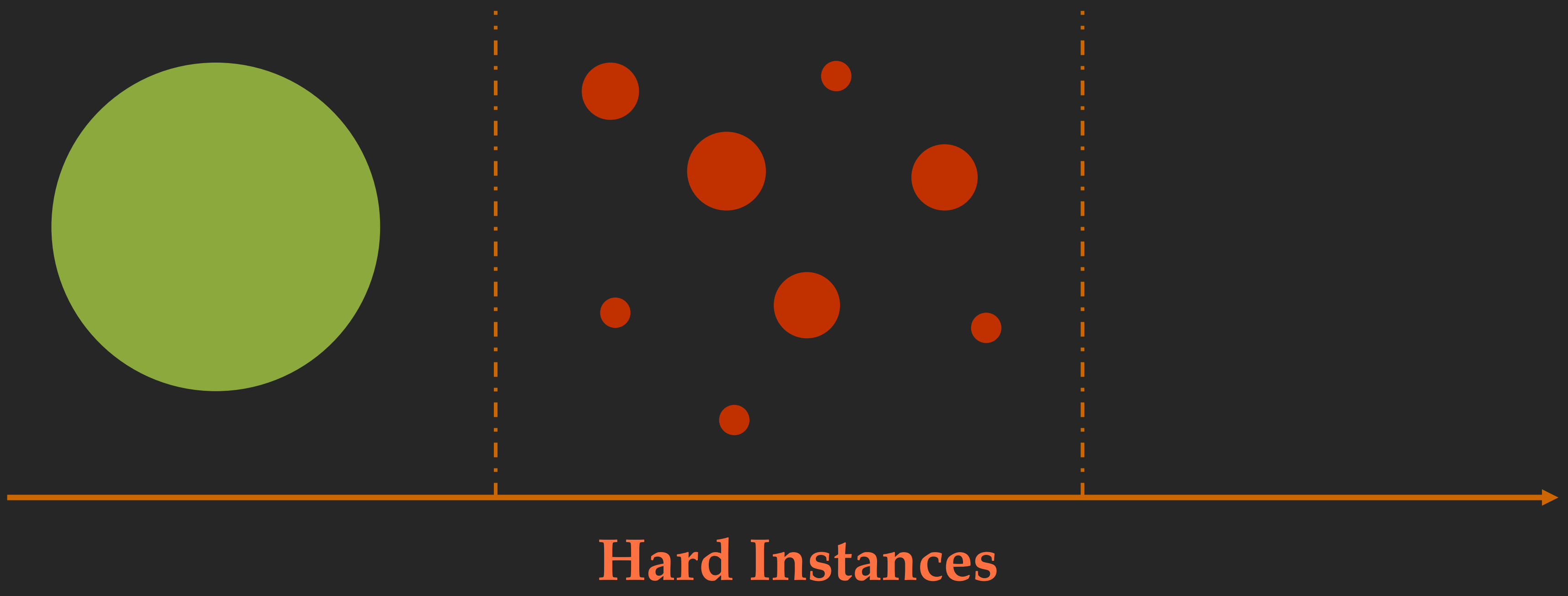
Average-Case Hardness



…but does not help for cryptographic utopia.

Worst-to-Average Case Reduction

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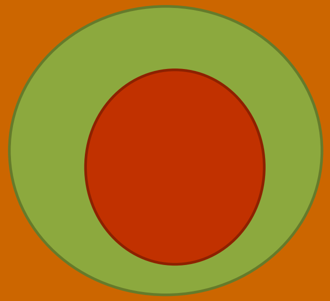
Average Case Hardness

Exists a distribution D over instances such that if we sample x from D, then x is hard with probability 0.5.



Worst-to-Average Case Reduction

worst-case problem
e.g. 3-SAT

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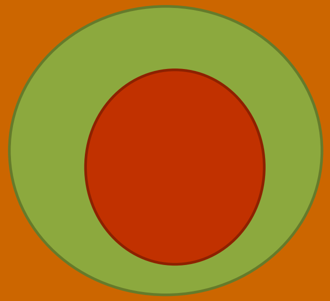

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we know problems that admit worst-to-average case reductions!

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Can we achieve Cryptographic Utopia?

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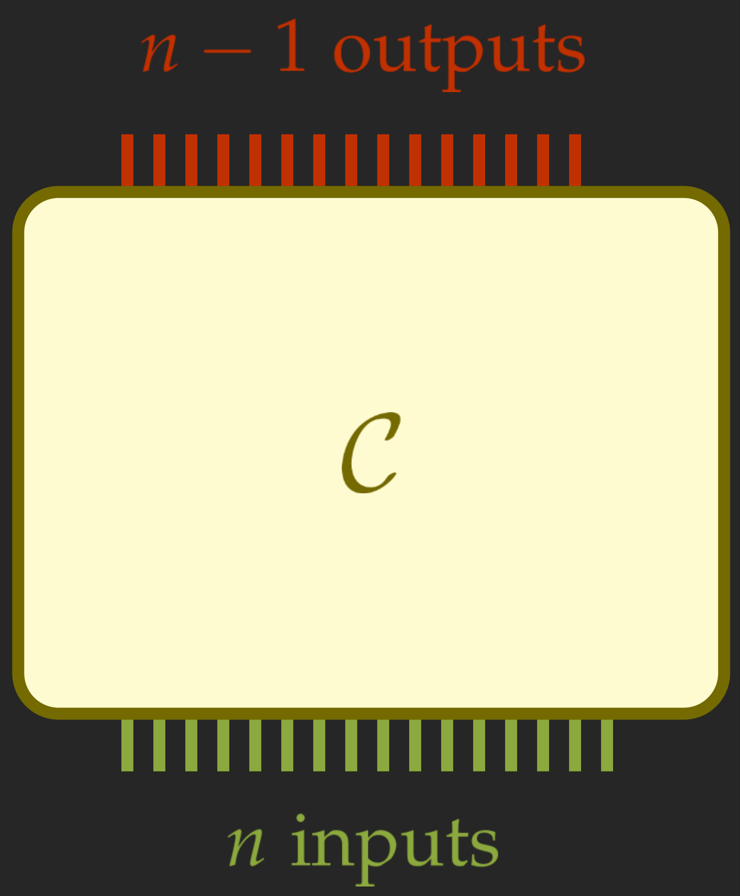
The Inadequacy of NP-hardness

Collision Resistant Hash Functions



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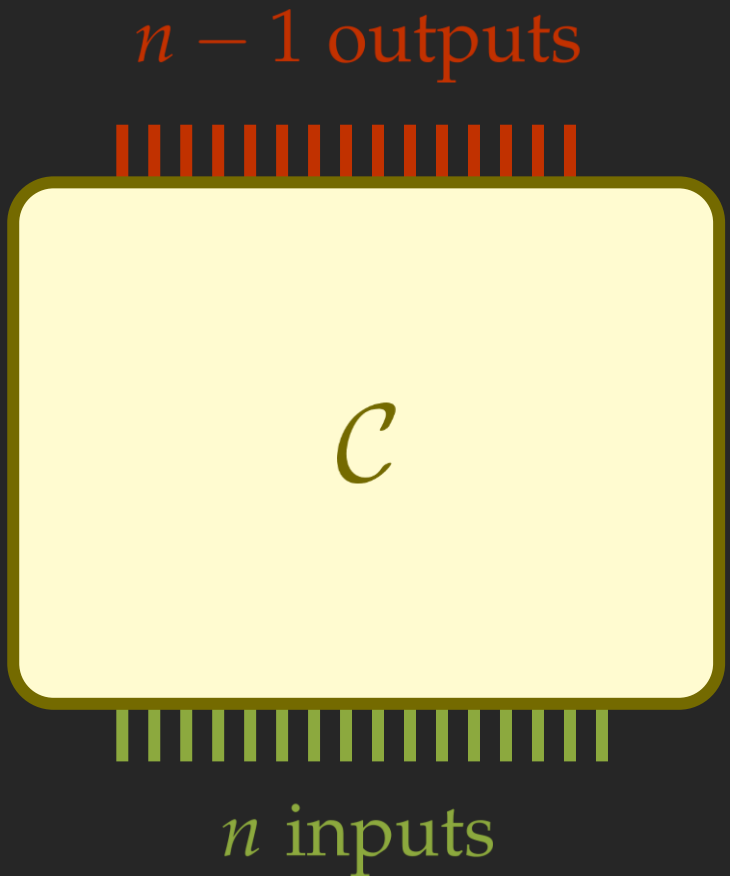

Hard to find x, x', with $x \neq x'$ and $\mathcal{C}(x) = \mathcal{C}(x')$

The Inadequacy of NP-hardness

To achieve cryptographic utopia for Collision Resistant Hash Functions we have to prove hardness for **search** problems that are **total**!

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Total Search Problem: the answer to the decision version of the problem is always affirmative, i.e. solution is guaranteed to exist.

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Total Search Problem: the answer to the decision version of the problem is always affirmative, i.e. solution is guaranteed to exist.

e.g. Any compressing function always has a collision!

The Inadequacy of NP-hardness

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Theorem **[Johnson Papadimitriou Yannakakis '88, Meggido Papadimitirou '91]**
If a total search problem is NP-hard then NP = co-NP.

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  PH collapses directly.

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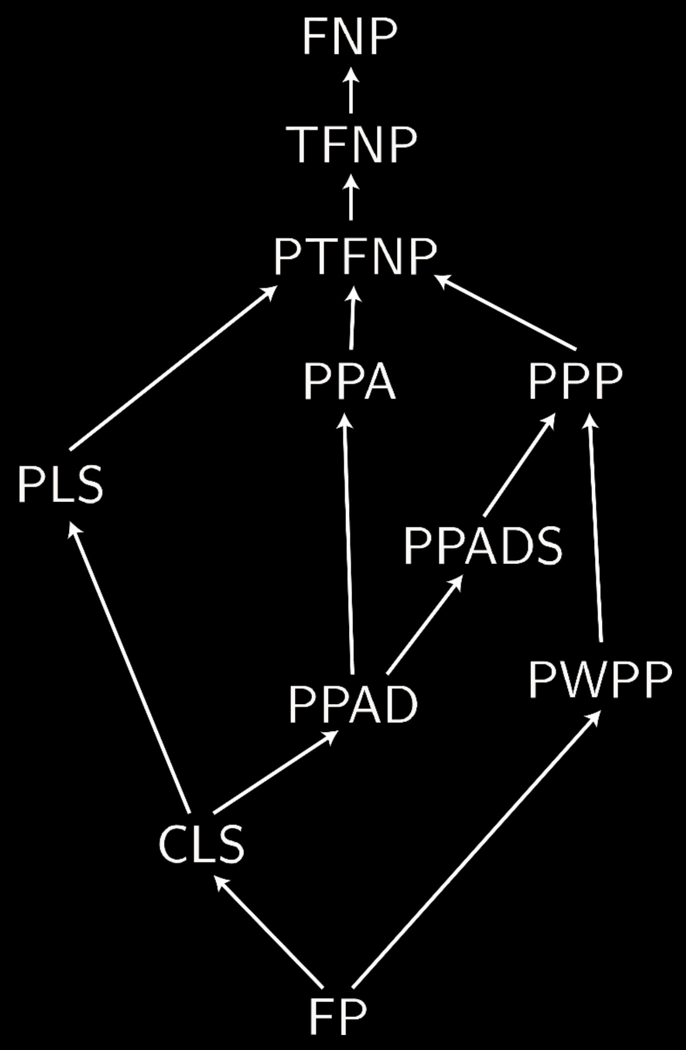
We cannot hope to use NP-hardness!

Complexity of Total Search Problems

FNP: class of search problems whose decision version is in NP.

TFNP: class of total search problems of FNP, i.e. a solution always exists [MP91]

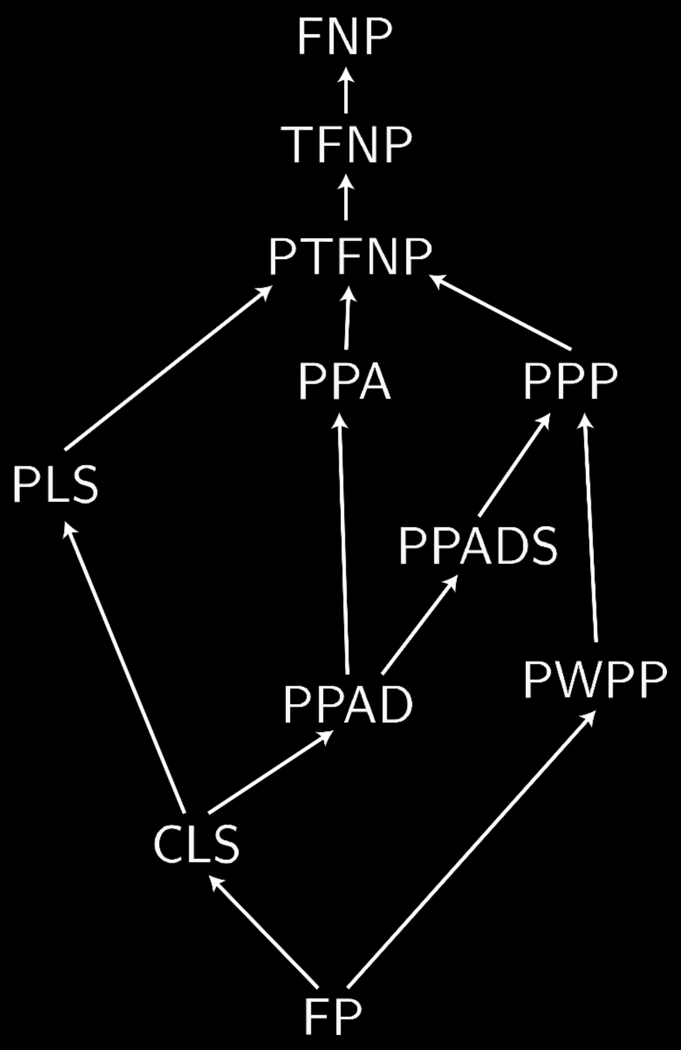
Subclasses of TFNP introduced by [JPY88, **Pap94**, CD11, Jerabek16]



Complexity of Total Search Problems

Many applications in game theory, economics, social choice, (discrete / continuous) optimization,

e.g. [JYP88], [BCE+98], [EGG06], [CDDT09], [DP11], [R15], [R16], [BIQ+17], [GP17], [DTZ18], [FG18] …



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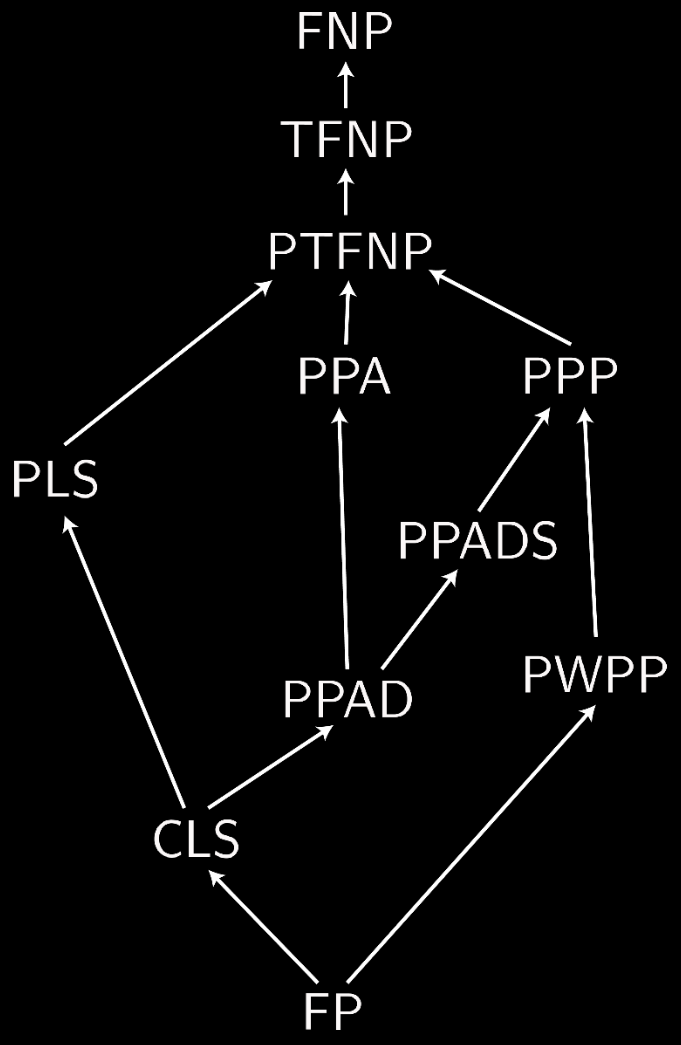
Most celebrated result:

 NASH is PPAD-complete [DGP06], [CDT06]



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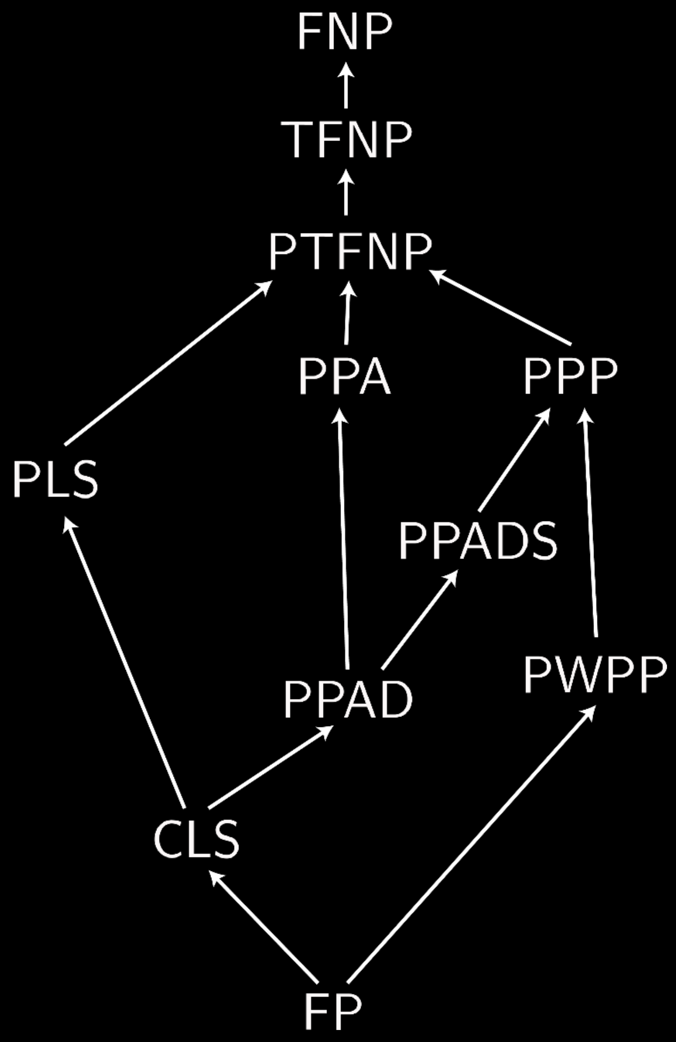
Connections to Cryptography:
[Bur06], [BPR15], [Jer16], [GPS16], [HY17], [RSS17], [HNY17], [KNY17]



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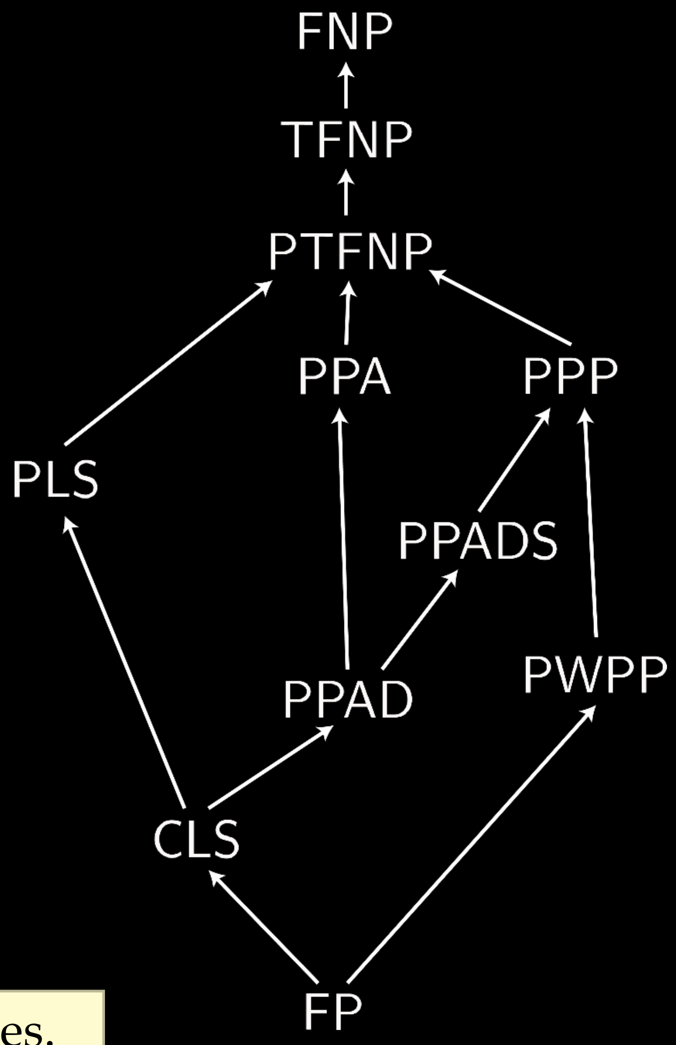
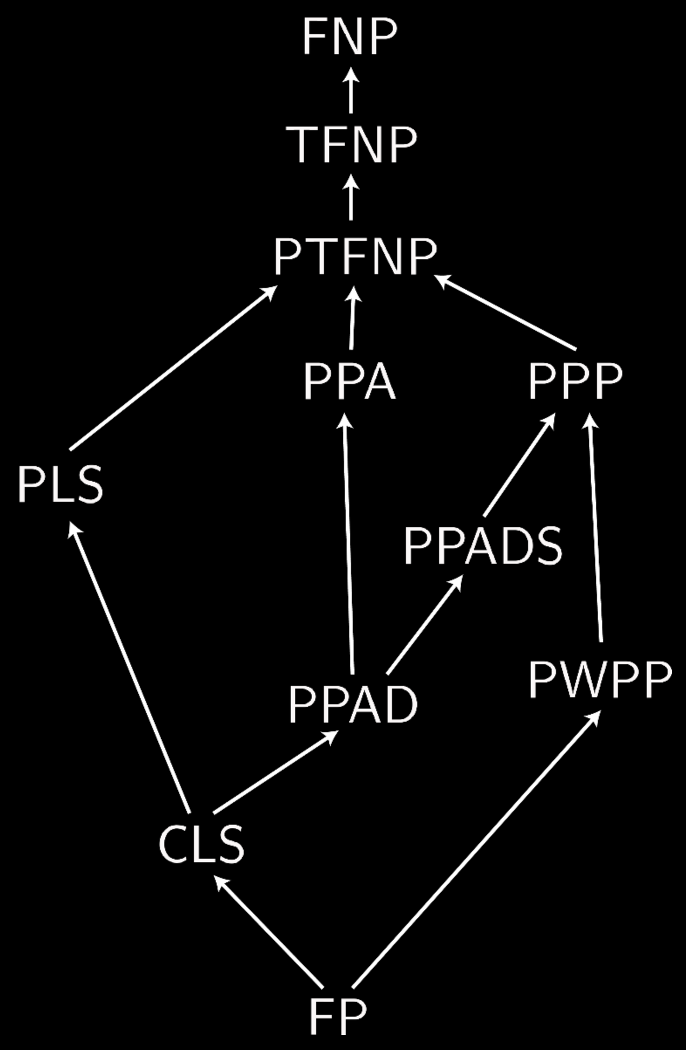


You can visit FOCS 2018 workshop on TFNP for references.

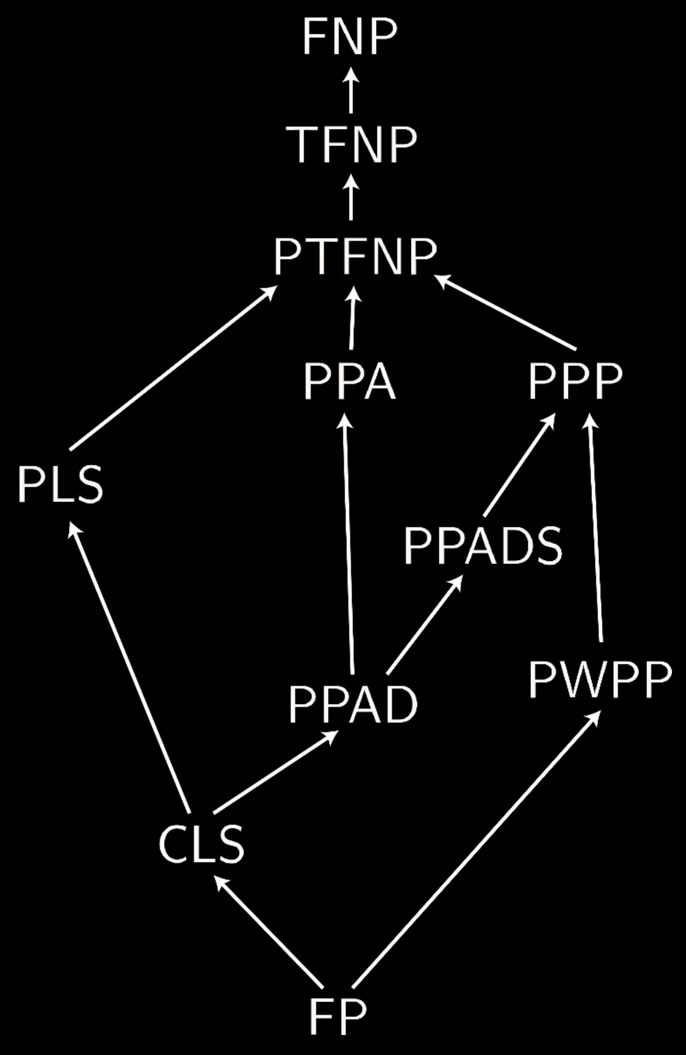
Complexity of Total Search Problems

Prior to our work **natural** complete problems for all subclasses except: PPP, PWPP, CLS, PPADS.



Complexity of Total Search Problems

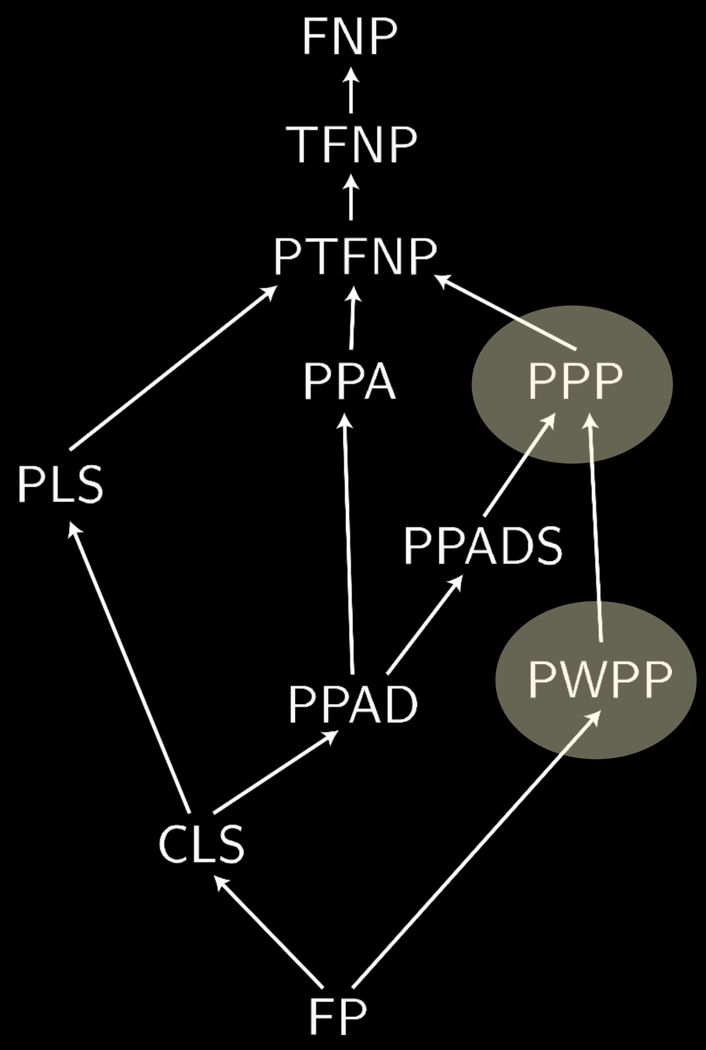
Natural: the problem does not contain a circuit or a Turing machine as part of the input.



Complexity of Total Search Problems

Prior to our work natural complete problems for all subclasses except: PPP, PWPP, CLS, PPADS.

Our Result
We identify the first natural PPP-complete and PWPP-complete problems answering an open problem since [Pap94].



Polynomial Pigeonhole Principle

"Total search problems should be classified in terms of the profound mathematical principles that are invoked to establish their totality."

Papadimitriou '94

PPP, PWPP $\longrightarrow$ Pigeonhole principle

Polynomial Pigeonhole Principle

PPP:

Given a circuit $C : \{0,1\}^n \to \{0,1\}^n$. Find:

Polynomial Pigeonhole Principle

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Given a circuit $C : \{0,1\}^n \to \{0,1\}^n$. Find:

1. An $\mathbf{x}$ s.t. $C(\mathbf{x}) = \mathbf{0}$ or
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Polynomial Pigeonhole Principle

PPP:
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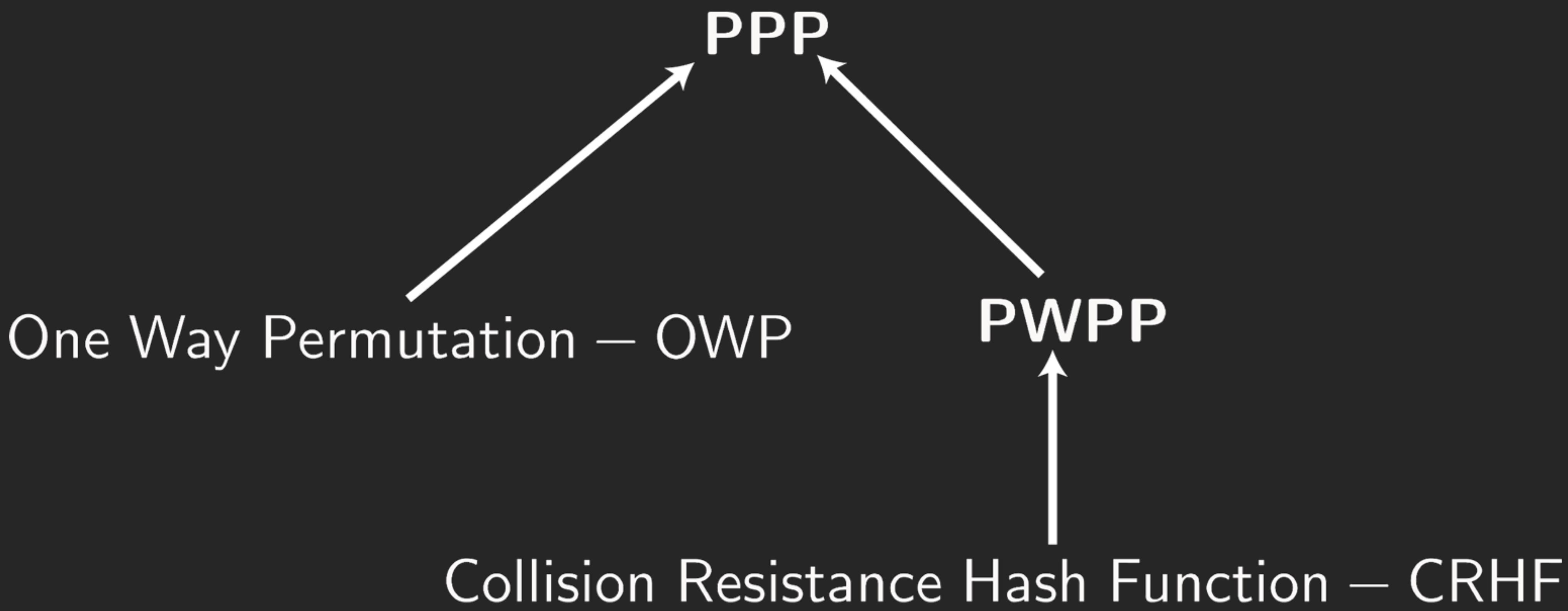
2. a collision, i.e $\mathbf{x} \neq \mathbf{y}$ s.t. $\mathsf{C}(\mathbf{x}) = \mathsf{C}(\mathbf{y})$.

Obviously a total problem, cannot be NP-hard!

Polynomial Pigeonhole Principle

PWPP:
Given a circuit $C : \{0,1\}^n \to \{0,1\}^m$, with $m < n$.
Find a collision, i.e $\mathbf{x} \neq \mathbf{y}$ s.t. $C(\mathbf{x}) = C(\mathbf{y})$.

PPP/PWPP and Cryptography



PPP/PWPP-completeness

A longstanding open problem since [Papadimitriou '94].

Our contribution:
We identify the first natural PPP/PWPP-complete problems.

This talk: **PWPP**.

Main Theorem:
WEAK-cSIS is PWPP-complete.

Short Integer Solution (SIS) Problem

INPUT: $\boxed{\mathbf{A}} \in \mathbb{Z}_q^{r \times m}$, with $m > \log(q)r$.

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OUTPUT: $\mathbf{x}$ s.t. $\|\mathbf{x}\| \leq 1$, $\mathbf{A}\mathbf{x} = \mathbf{0} \pmod q$

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domain size is 2^m

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image size is q^r

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INPUT: $\mathbf{A} \in \mathbb{Z}_q^{r \times m}$, with $2^m > q^r$.

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Short Integer Solution (SIS) Problem

The problem is total!

INPUT: $\mathbf{A} \in \mathbb{Z}_q^{r \times m}$, with $2^m > q^r$.

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Short Integer Solution (SIS) Problem

The problem is in PWPP!

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Constraint Short Integer Solution Problem

INPUT: $\mathbf{A} \in \mathbb{Z}_q^{r \times m}$, with $m > \log(q)(r + d)$ $\mathbf{G} \in \mathbb{Z}_q^{d \times m}$, and *binary invertible*

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$$\mathbf{G} = \begin{bmatrix} \mathbf{g} & & & \star & \star & \star \\ & \mathbf{g} & & & & \\ \mathbf{0} & & \mathbf{g} & & & \end{bmatrix}$$

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Binary Invertible Matrix

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e.g. for m = 10, q = 8

$$\mathbf{G} = \begin{bmatrix} 1 & 2 & 4 & 3 & 0 & 6 & 5 & 6 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 4 & 1 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 0 \end{bmatrix}$$

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$$\mathbf{g} = \begin{bmatrix} 1 & 2 & 4 & \ldots & 2^{k-1} \end{bmatrix} \qquad 2^k \geq q$$

e.g. for m = 10, q = 8

$$\mathbf{G} = \begin{bmatrix} 1 & 2 & 4 & 3 & 0 & 6 & 5 & 6 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 4 & 1 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 0 \end{bmatrix}$$

Binary Invertible Matrix

Lemma
For any $\mathbf{z} \in \{0,1\}^{m-\log(q)d}$ and any $\mathbf{b} \in \mathbb{Z}_q^d$, we can efficiently compute a binary solution of the form $\mathbf{x} = [\star \;\; \star \cdots \star \;\; \mathbf{z}]$ for the equation $\mathbf{G}\mathbf{x} = \mathbf{b} \pmod q$.

Binary Invertible Matrix

Example

$$\begin{bmatrix} 1 & 2 & 4 & 3 & 0 & 6 & 5 & 6 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 4 & 1 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 0 \end{bmatrix} \begin{bmatrix} \star \\ \star \\ \star \\ \star \\ \star \\ \star \\ \star \\ \star \\ \star \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \pmod 8$$

Binary Invertible Matrix

Example

$$\begin{bmatrix} 1 & 2 & 4 & 3 & 0 & 6 & 5 & 6 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 4 & 1 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 0 \end{bmatrix} \begin{bmatrix} \star \\ \star \\ \star \\ \star \\ \star \\ \star \\ \star \\ \star \\ \star \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \pmod 8$$

Binary Invertible Matrix

Example

$$\begin{bmatrix} 1 & 2 & 4 & 3 & 0 & 6 & 5 & 6 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 4 & 1 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 0 \end{bmatrix} \begin{bmatrix} \star \\ \star \\ \star \\ \star \\ \star \\ \star \\ \star \\ \star \\ \star \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \pmod 8$$

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Binary Invertible Matrix

Example

$$\begin{bmatrix} 1 & 2 & 4 & 3 & 0 & 6 & 5 & 6 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 4 & 1 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & \boxed{2} & \boxed{4} & 0 \end{bmatrix} \begin{bmatrix} \star \\ \star \\ \star \\ \star \\ \star \\ \star \\ \boxed{x_7} \\ \boxed{x_8} \\ \boxed{x_9} \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \pmod 8$$

Binary Invertible Matrix

Example

$$\begin{bmatrix} 1 & 2 & 4 & 3 & 0 & 6 & 5 & 6 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 4 & 1 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1 \quad 2 \quad 4} & & & 0 \end{bmatrix} \begin{bmatrix} \star \\ \star \\ \star \\ \star \\ \star \\ \star \\ x_7 \\ x_8 \\ x_9 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \pmod 8$$

$$1 \cdot x_7 + 2 \cdot x_8 + 4 \cdot x_9 = 1 \pmod 8$$

Binary Invertible Matrix

Example

$$\begin{bmatrix} 1 & 2 & 4 & 3 & 0 & 6 & 5 & 6 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 4 & 1 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1 \quad 2 \quad 4} & & & 0 \end{bmatrix} \begin{bmatrix} \star \\ \star \\ \star \\ \star \\ \star \\ \star \\ \boxed{1} \\ \boxed{0} \\ \boxed{0} \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \pmod 8$$

Binary Invertible Matrix

Example

$$\begin{bmatrix} 1 & 2 & 4 & 3 & 0 & 6 & 5 & 6 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 4 & 1 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 0 \end{bmatrix} \begin{bmatrix} \star \\ \star \\ \star \\ \star \\ \star \\ \star \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \pmod 8$$

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Binary Invertible Matrix

Example

$$\begin{bmatrix} 1 & 2 & 4 & 3 & 0 & 6 & 5 & 6 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 4 & 1 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 0 \end{bmatrix} \begin{bmatrix} \star \\ \star \\ \star \\ x_4 \\ x_5 \\ x_6 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \pmod 8$$

Binary Invertible Matrix

Example

$$\begin{bmatrix} 1 & 2 & 4 & 3 & 0 & 6 & 5 & 6 & 2 & 1 \\ 0 & 0 & 0 & \boxed{1 \quad 2 \quad 4} & & & 1 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 0 \end{bmatrix} \begin{bmatrix} \star \\ \star \\ \star \\ x_4 \\ x_5 \\ x_6 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \pmod 8$$

Binary Invertible Matrix

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Binary Invertible Matrix

Example

$$\begin{bmatrix} 1 & 2 & 4 & 3 & 0 & 6 & 5 & 6 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 4 & 1 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 0 \end{bmatrix} \begin{bmatrix} \star \\ \star \\ \star \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \pmod{8}$$

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$$\begin{bmatrix} \boxed{1 & 2 & 4} & 3 & 0 & 6 & 5 & 6 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 4 & 1 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \pmod 8$$

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Example

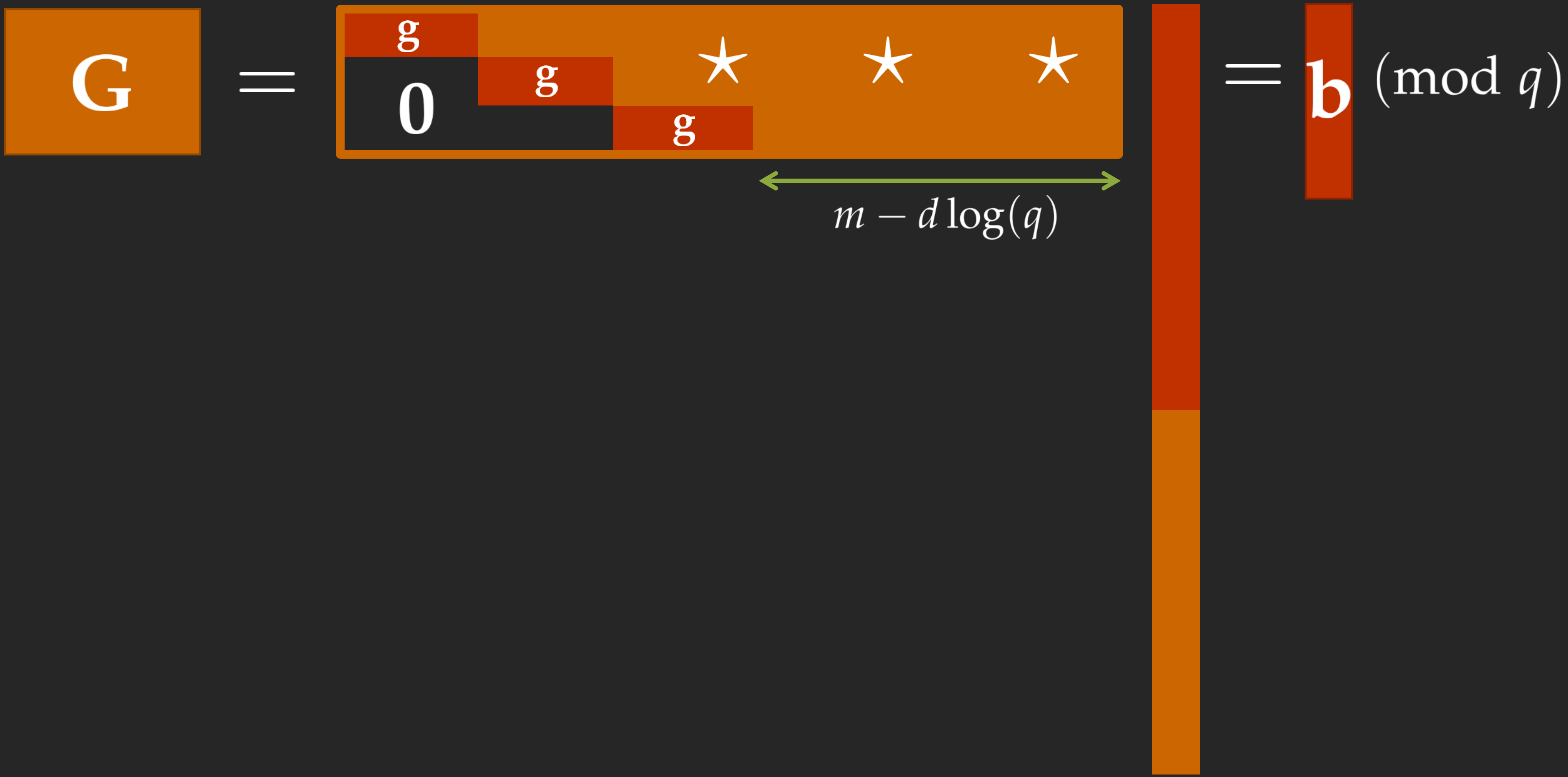
$$\begin{bmatrix} \boxed{1 & 2 & 4} & 3 & 0 & 6 & 5 & 6 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 4 & 1 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \pmod 8$$

Binary Invertible Matrix

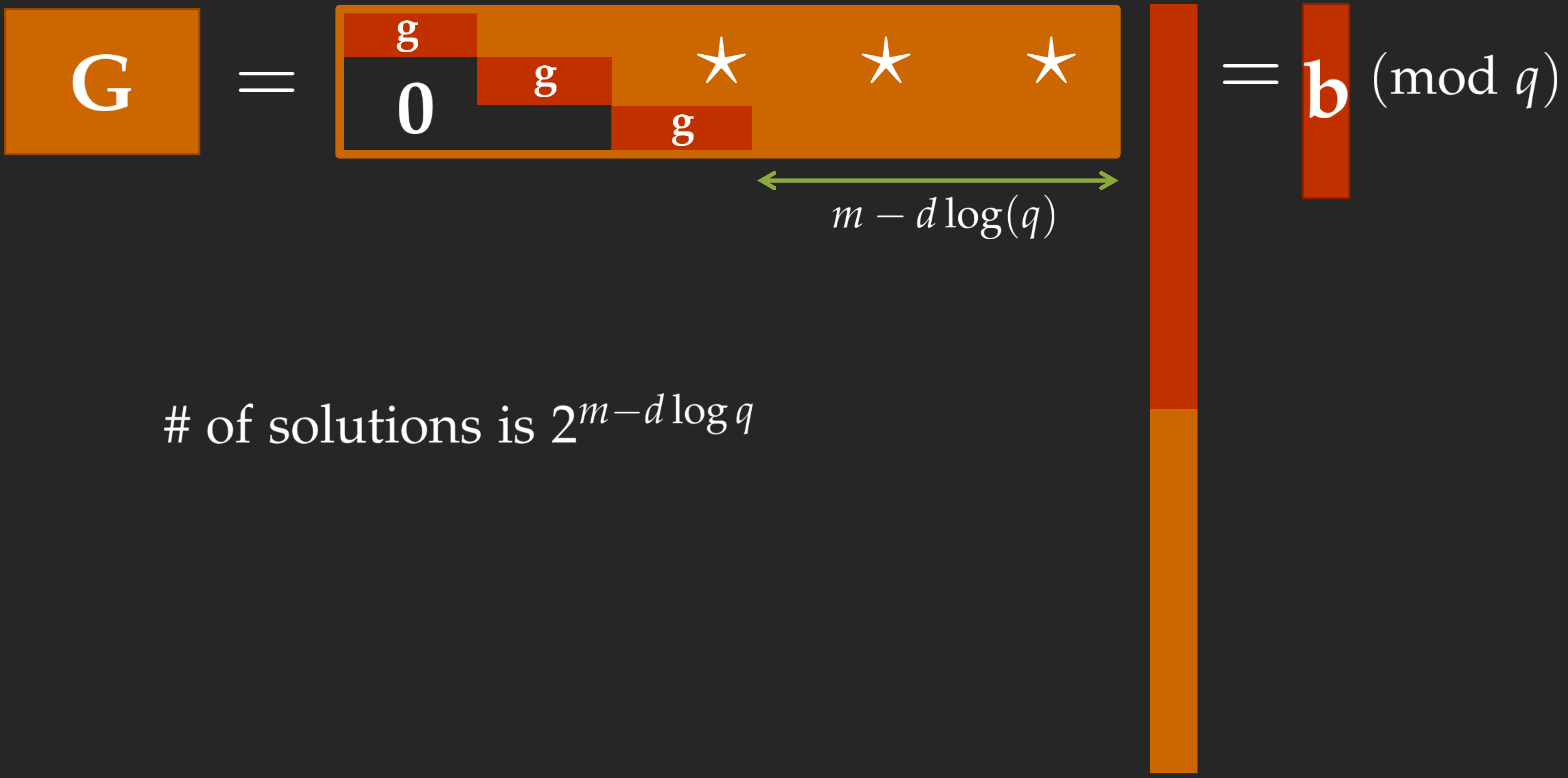
Example

$$\begin{bmatrix} 1 & 2 & 4 & 3 & 0 & 6 & 5 & 6 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 4 & 1 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \pmod 8$$

WEAK-cSIS is Total



WEAK-cSIS is Total



# of solutions is $2^{m-d\log q}$

WEAK-cSIS is Total

INPUT: $\mathbf{A} \in \mathbb{Z}_q^{r\times m}$, with $m > \log(q)(r+d)$ $\quad \mathbf{G} \in \mathbb{Z}_q^{d\times m}$, and *binary invertible*

OUTPUT: $\mathbf{x}\ \mathbf{y} \in \{0,1\}^m$ s.t. $\mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{y} \pmod q$

$$\mathbf{G}\mathbf{x} = \mathbf{G}\mathbf{y} = \mathbf{0} \pmod q$$

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WEAK-cSIS is Total

INPUT: $\mathbf{A} \in \mathbb{Z}_q^{r \times m}$, with $m - d\log(q) > r\log(q)$ $\quad \mathbf{G} \in \mathbb{Z}_q^{d \times m}$, and *binary invertible*

OUTPUT: $\mathbf{x}\ \mathbf{y} \in \{0,1\}^m$ s.t. $\mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{y} \pmod q$

$\mathbf{G}\mathbf{x} = \mathbf{G}\mathbf{y} = \mathbf{0} \pmod q$

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$\blacksquare$

WEAK-CSIS is in PWPP

Lemma
For any $\mathbf{z} \in \{0,1\}^{m-\log(q)d}$ and any $\mathbf{b} \in \mathbb{Z}_q^d$, we can efficiently compute a binary solution of the form $\mathbf{x} = [\star \;\; \star \cdots \star \;\; \mathbf{z}]$ for the equation $\mathbf{G}\mathbf{x} = \mathbf{b} \pmod q$.

WEAK-cSIS is in PWPP

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Since $m > (r+d)\log(q)$, there exist more than $2^{\log(q)r} = q^r$, $\mathbf{x} \in \{0,1\}^m$ such that $\mathbf{Gx} = \mathbf{b} \pmod q$.

WEAK-cSIS is in PWPP

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For any $\mathbf{z} \in \{0,1\}^{m-\log(q)d}$ and any $\mathbf{b} \in \mathbb{Z}_q^d$, we can efficiently compute a binary solution of the form $\mathbf{x} = [\star \;\; \star \cdots \star \;\; \mathbf{z}]$ for the equation $\mathbf{G}\mathbf{x} = \mathbf{b} \pmod q$.

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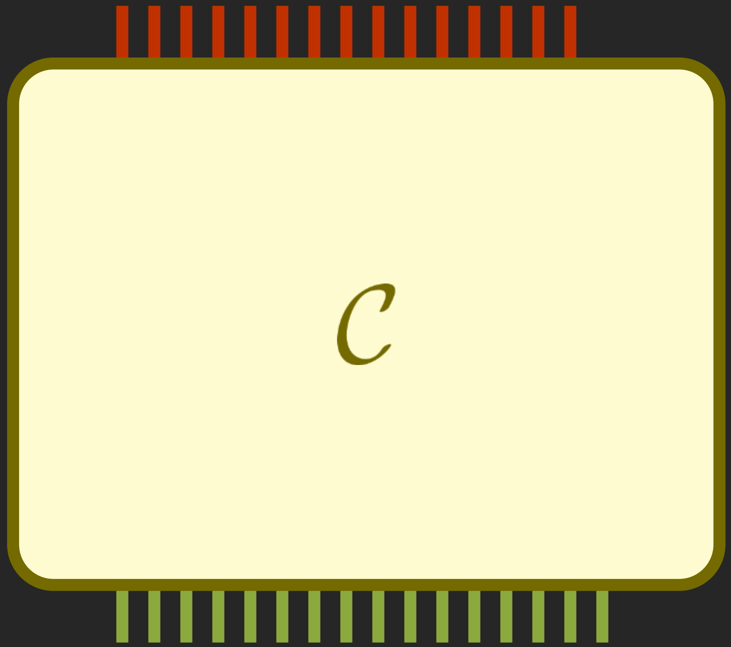
There exist $\mathbf{x} \neq \mathbf{y}$ such that $\mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{y} \pmod q$ and $\mathbf{G}\mathbf{x} = \mathbf{G}\mathbf{y} = \mathbf{b} \pmod q$.

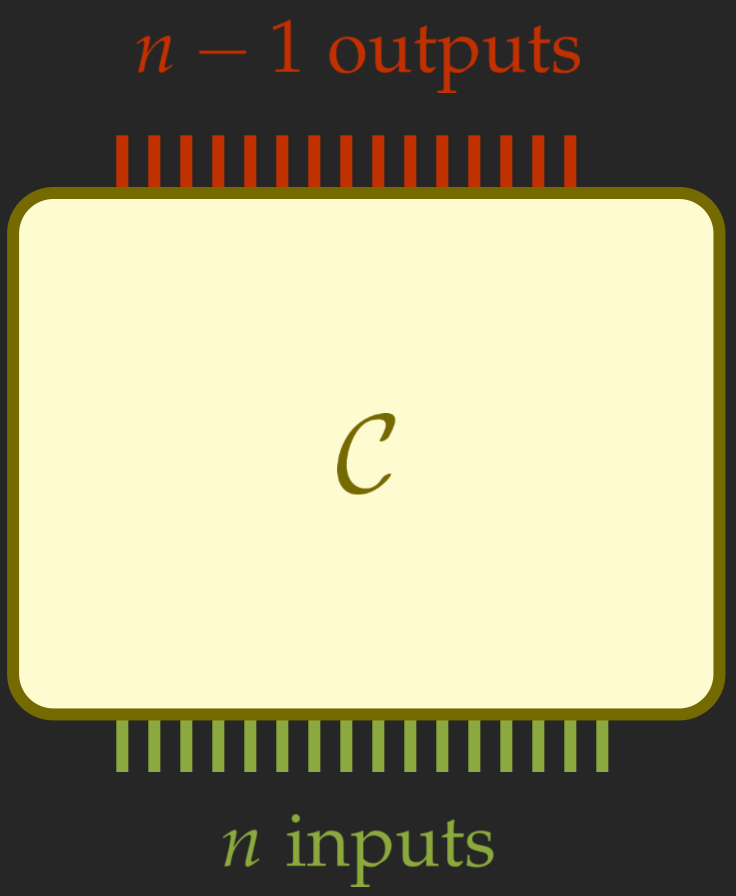
WEAK-CSIS is PWPP-hard

PWPP:
Given a circuit $C : \{0,1\}^n \to \{0,1\}^m$, with $m < n$.
Find a collision, i.e $\mathbf{x} \neq \mathbf{y}$ s.t. $C(\mathbf{x}) = C(\mathbf{y})$.

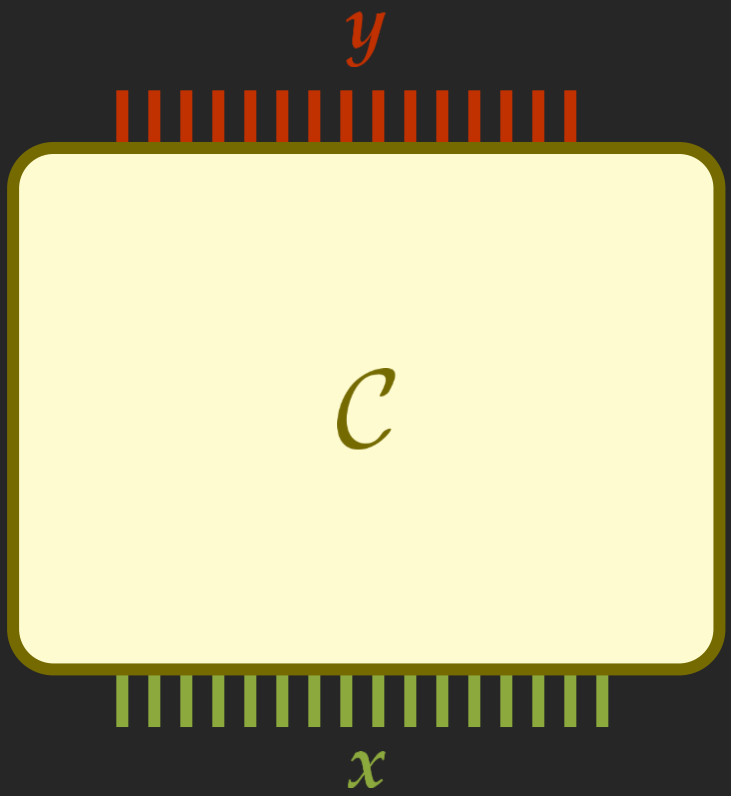
WEAK-CSIS is PWPP-hard



WEAK-cSIS is PWPP-hard



WEAK-cSIS is PWPP-hard



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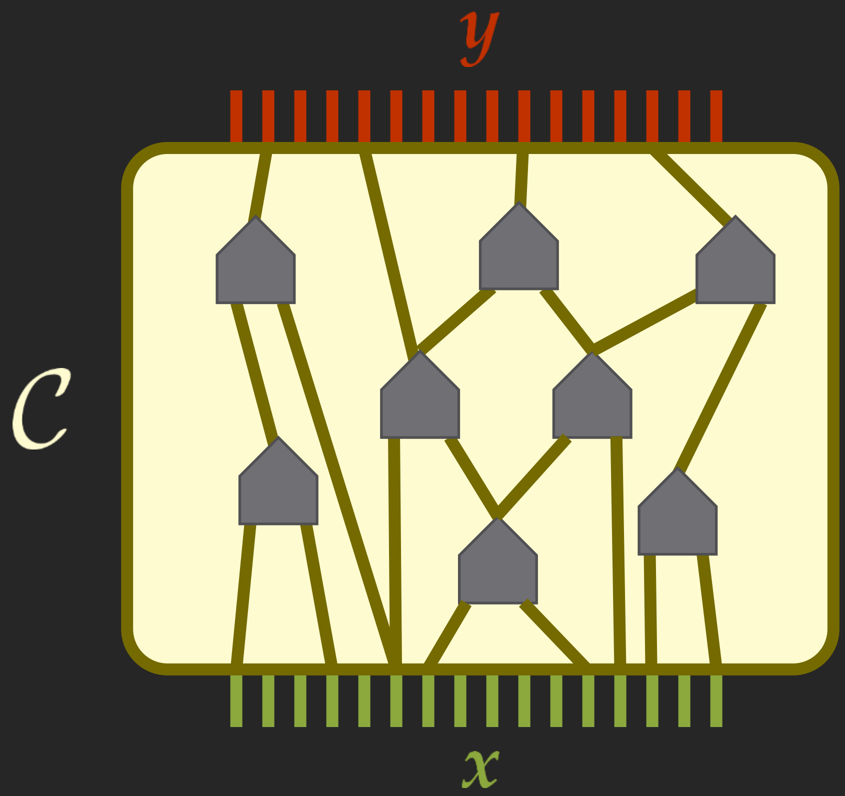

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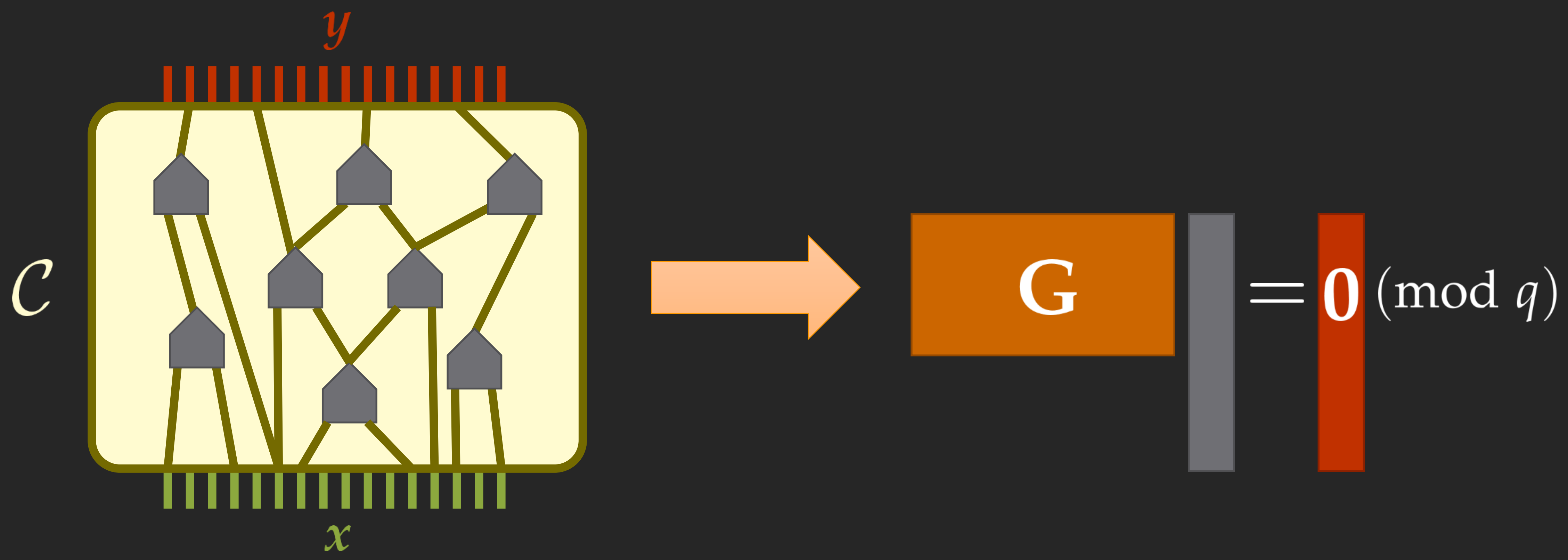

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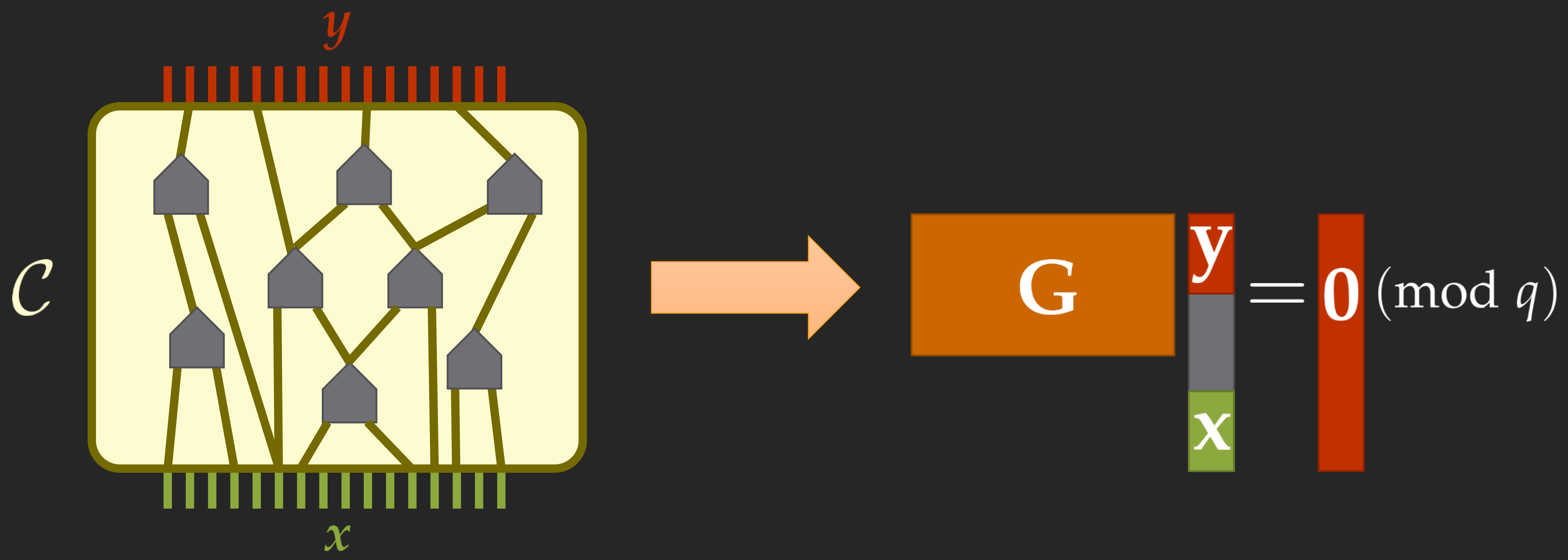

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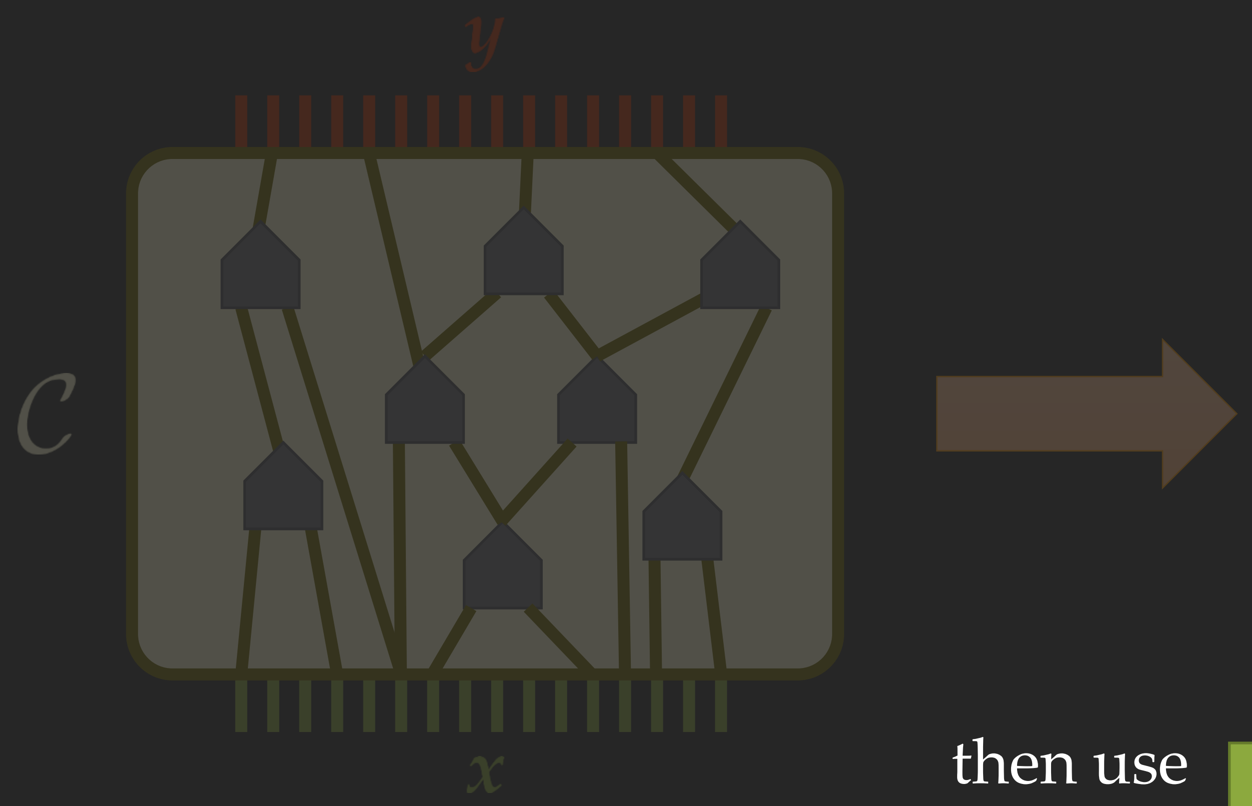
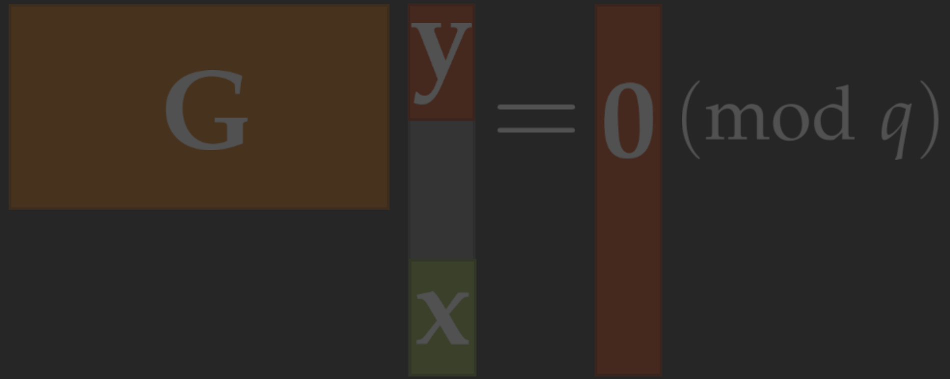
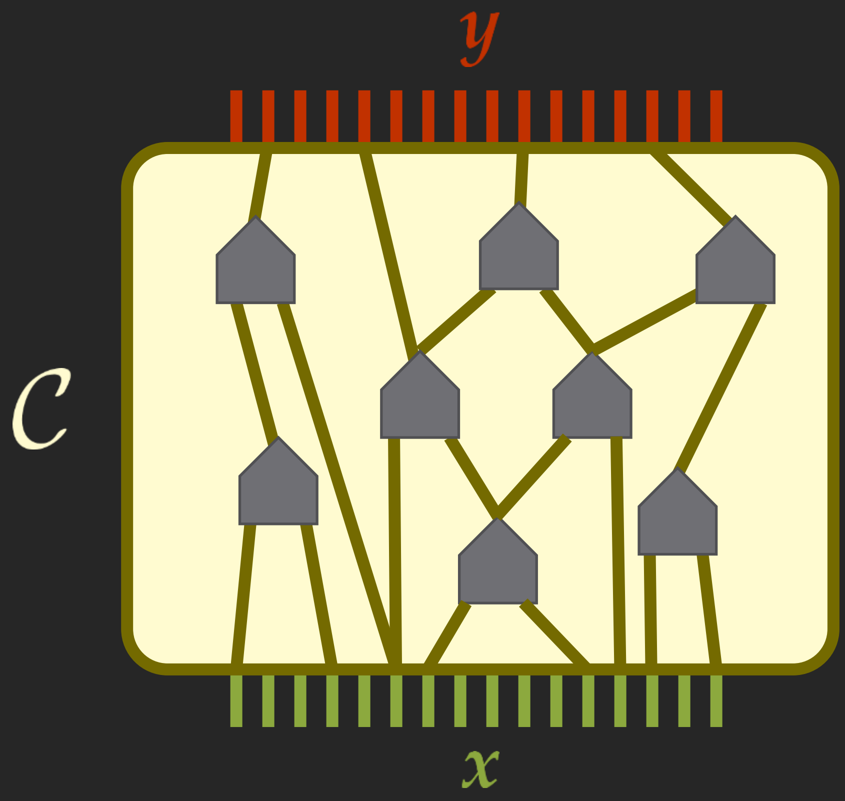




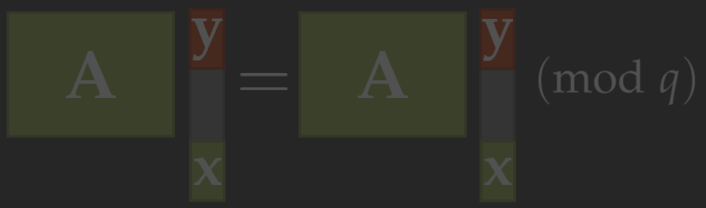
then use $\mathbf{A} \begin{bmatrix} \mathbf{y} \\ \mathbf{x} \end{bmatrix} = \mathbf{A} \begin{bmatrix} \mathbf{y} \\ \mathbf{x} \end{bmatrix} \pmod q$

to find a collision!

WEAK-cSIS is PWPP-hard



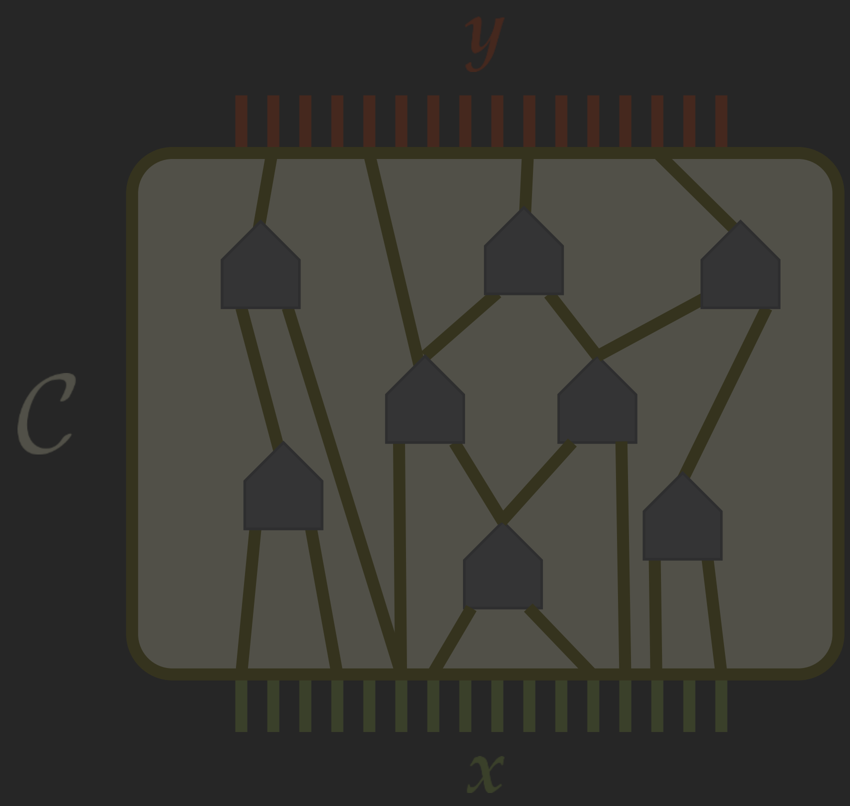
then use



to find a collision!

WEAK-cSIS is PWPP-hard

Attention!
During the reduction we have to preserve **totality**!



$$\mathbf{G}\begin{bmatrix}\mathbf{y}\\ \mathbf{x}\end{bmatrix} = \mathbf{0} \pmod{q}$$

then use $\mathbf{A}\begin{bmatrix}\mathbf{y}\\ \mathbf{x}\end{bmatrix} = \mathbf{A}\begin{bmatrix}\mathbf{y}\\ \mathbf{x}\end{bmatrix} \pmod{q}$

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WEAK-cSIS is PWPP-hard

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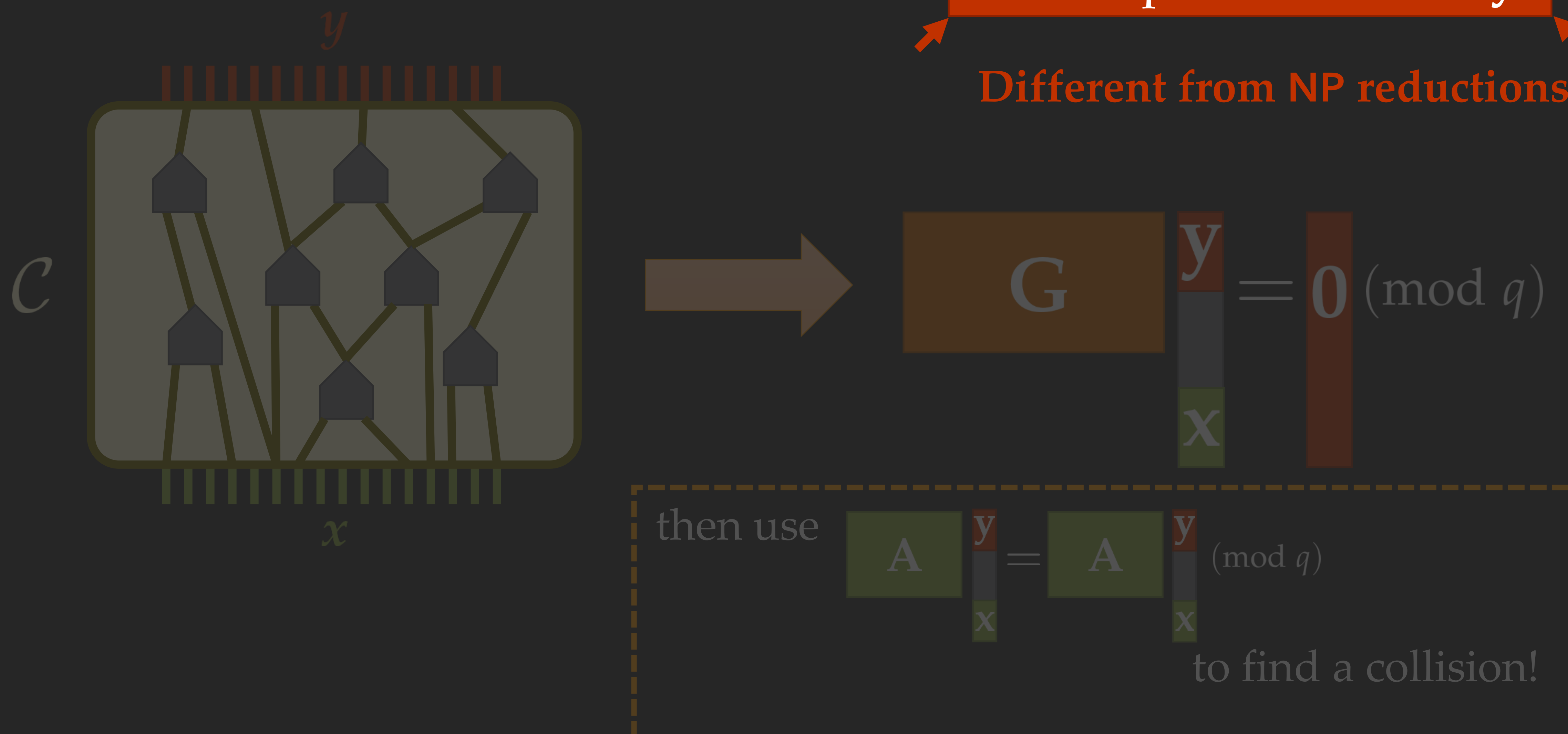
Different from NP reductions!



Hash Function from WEAK-CSIS

<u>Hash function family:</u>

- Key: $\mathbf{A} \in \mathbb{Z}_q^{r \times m}$, with $m > \log(q)(r+d)$ $\mathbf{G} \in \mathbb{Z}_q^{d \times m}$ binary invertible matrix

Hash Function from WEAK-CSIS

<u>Hash function family:</u>

- Key: $\mathbf{A} \in \mathbb{Z}_q^{r \times m}$, with $m > \log(q)(r+d)$ $\quad \mathbf{G} \in \mathbb{Z}_q^{d \times m}$ binary invertible matrix
- Hash(x): For $\mathbf{x} \in \{0,1\}^{m - d\log(q)}$, use Lemma to find $\mathbf{z} \in \{0,1\}^{d\log(q)}$ s.t. $\mathbf{G}\begin{bmatrix}\mathbf{z}\\\mathbf{x}\end{bmatrix} = \mathbf{0} \pmod q$.

$$\mathbf{A}\begin{bmatrix}\mathbf{z}\\\mathbf{x}\end{bmatrix} \pmod q$$

Can we achieve Cryptographic Utopia?

Bottlenecks

- cryptography is based on problems that are hard on average!
- NP-hard problems do not suffice for cryptography.

Can we achieve Cryptographic Utopia?

Bottlenecks

- **cryptography is based on problems that are hard on average!**
- NP-hard problems do not suffice for cryptography.

Approximate Short Integer Solution (APPROXSIS)

INPUT: $\mathbf{A} \in \mathbb{Z}_q^{r\times m}$, with $m > \log(q)r$.

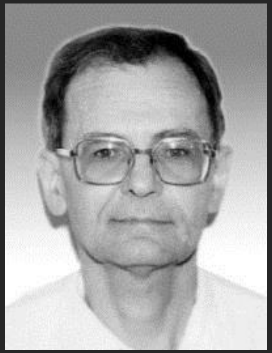
OUTPUT: $\mathbf{x}$ s.t. $\|\mathbf{x}\|_2 \leq B$, $\mathbf{A}\mathbf{x} = \mathbf{0} \pmod{q}$

Average Short Integer Solution (AverageSIS)

INPUT: $\boxed{\mathbf{A}} \sim U\left[\mathbb{Z}_q^{r\times m}\right]$, with $m > \log(q)r$.

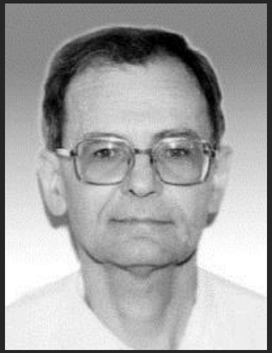
OUTPUT: $\mathbf{x}$ s.t. $\|\mathbf{x}\|_\infty \leq 1$, $\mathbf{A}\mathbf{x} = \mathbf{0} \pmod{q}$

Worst-to-Average Case Reduction for SIS



Informal Theorem[Ajtai'96]
There is a randomized Cook reduction from the **worst-case** problem APPROXSIS to the **average-case** problem AVERAGESIS!

Worst-to-Average Case Reduction for SIS



Informal Theorem [Ajtai'96]
There is a randomized Cook reduction from the **worst-case** problem APPROXSIS to the **average-case** problem AVERAGESIS!

This result is the foundation of lattice based cryptography.

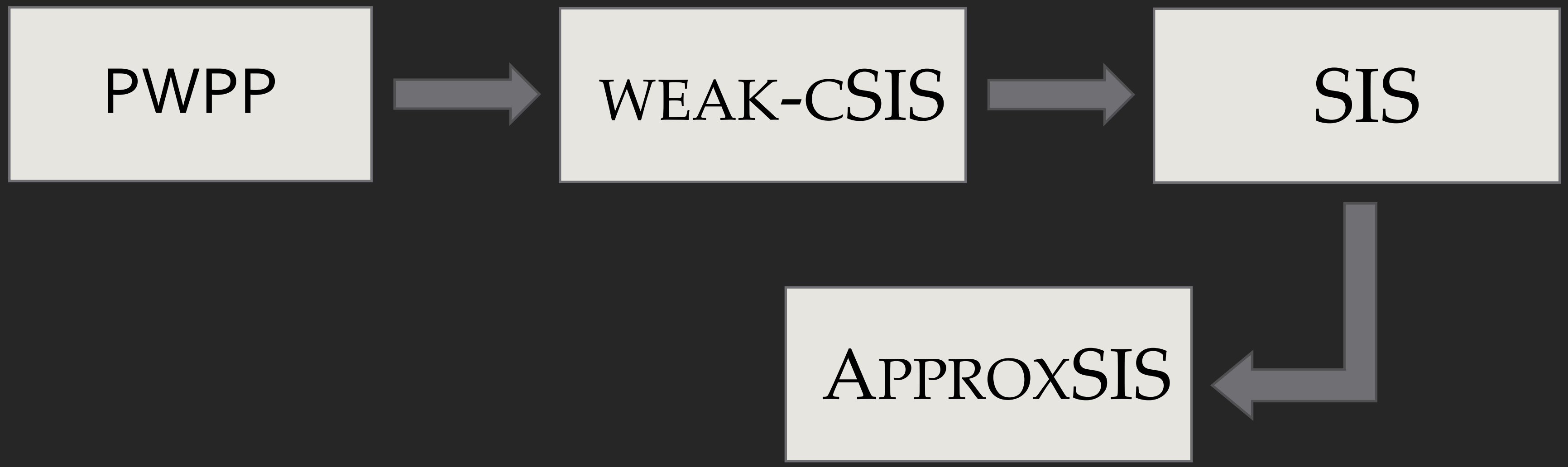
Can we achieve Cryptographic Utopia?

PWPP $\rightarrow$ WEAK-cSIS

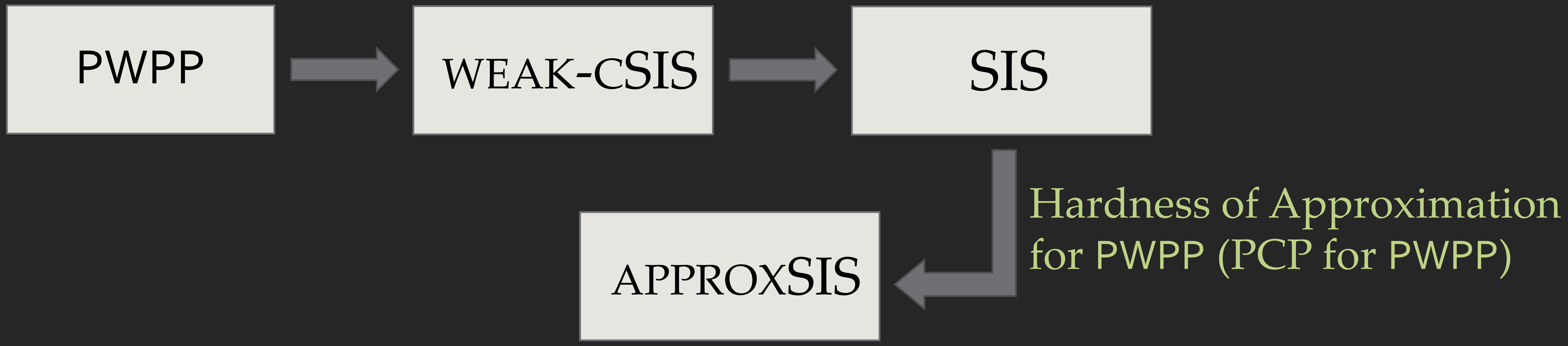
Can we achieve Cryptographic Utopia?

PWPP → WEAK-cSIS → SIS

Can we achieve Cryptographic Utopia?



Can we achieve Cryptographic Utopia?



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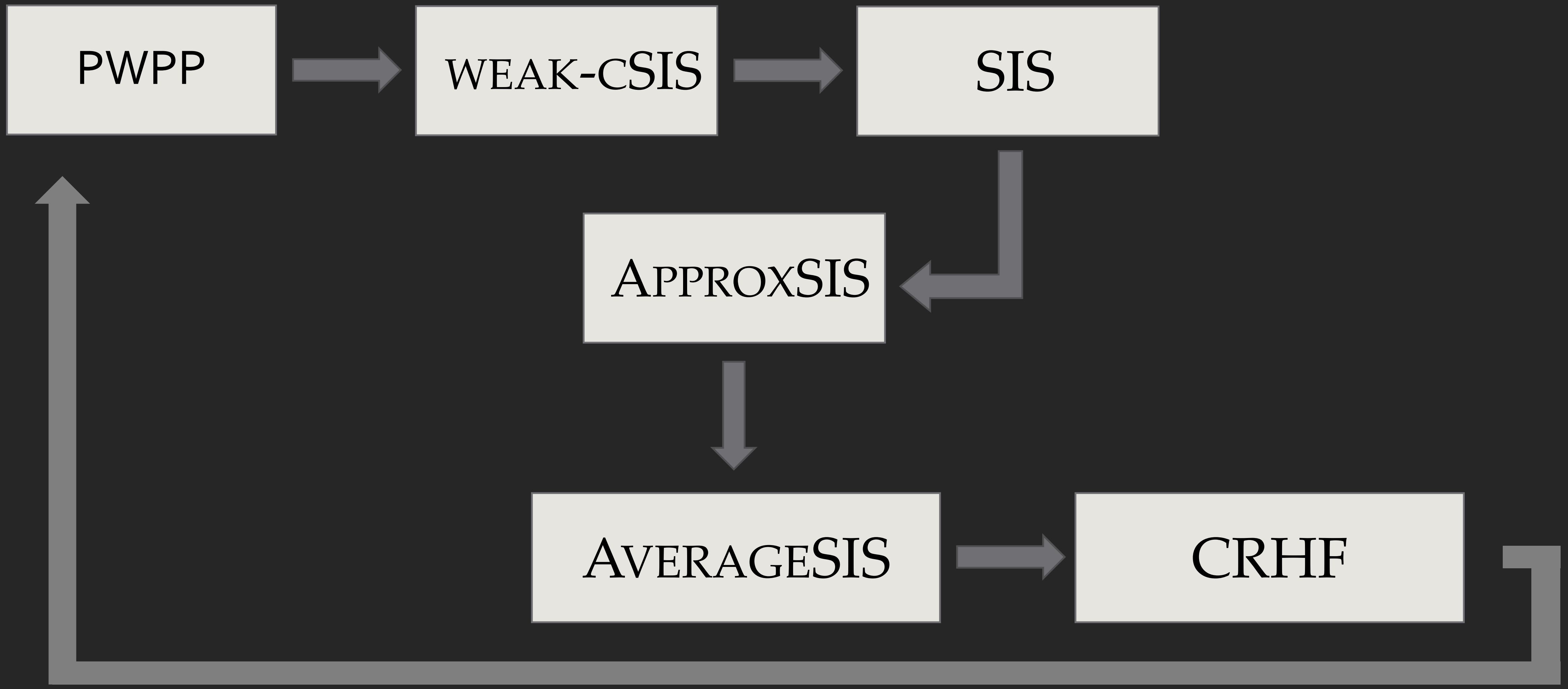

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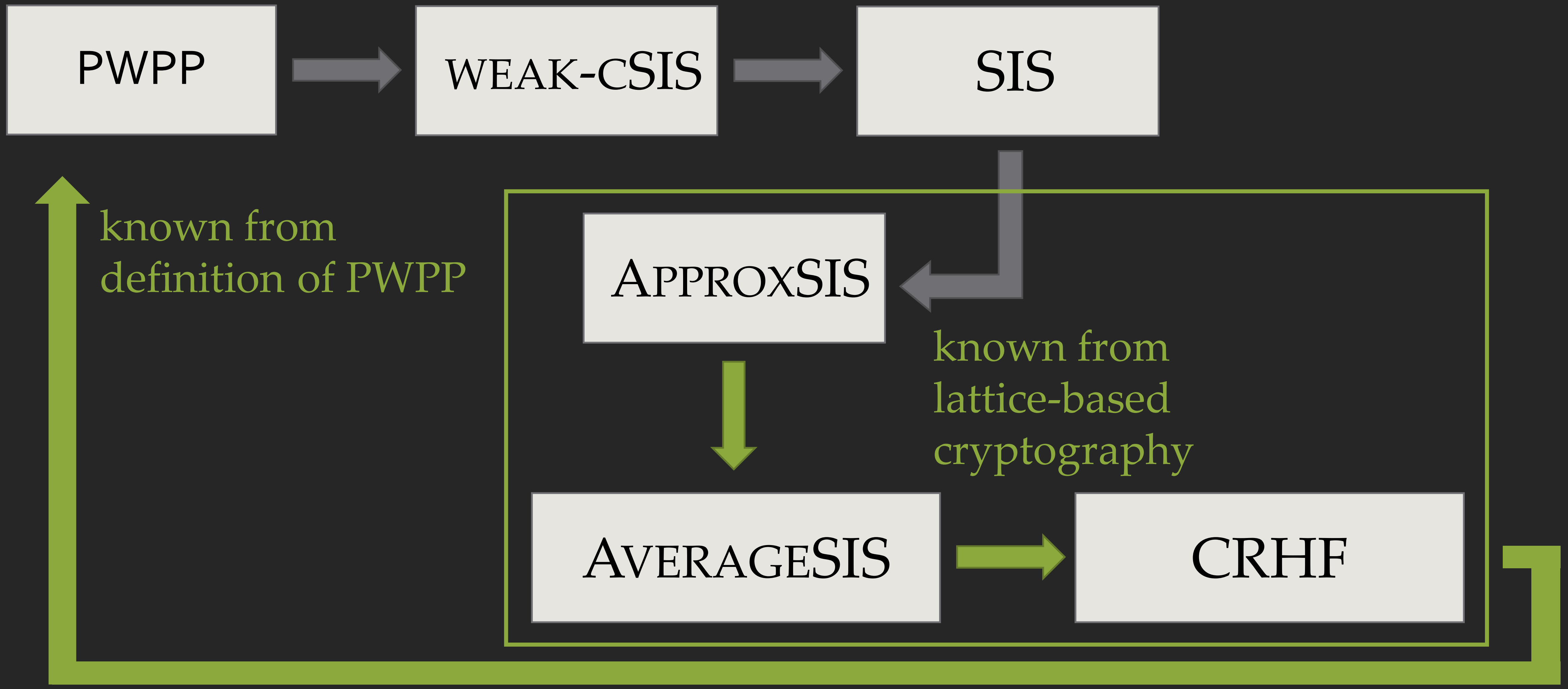

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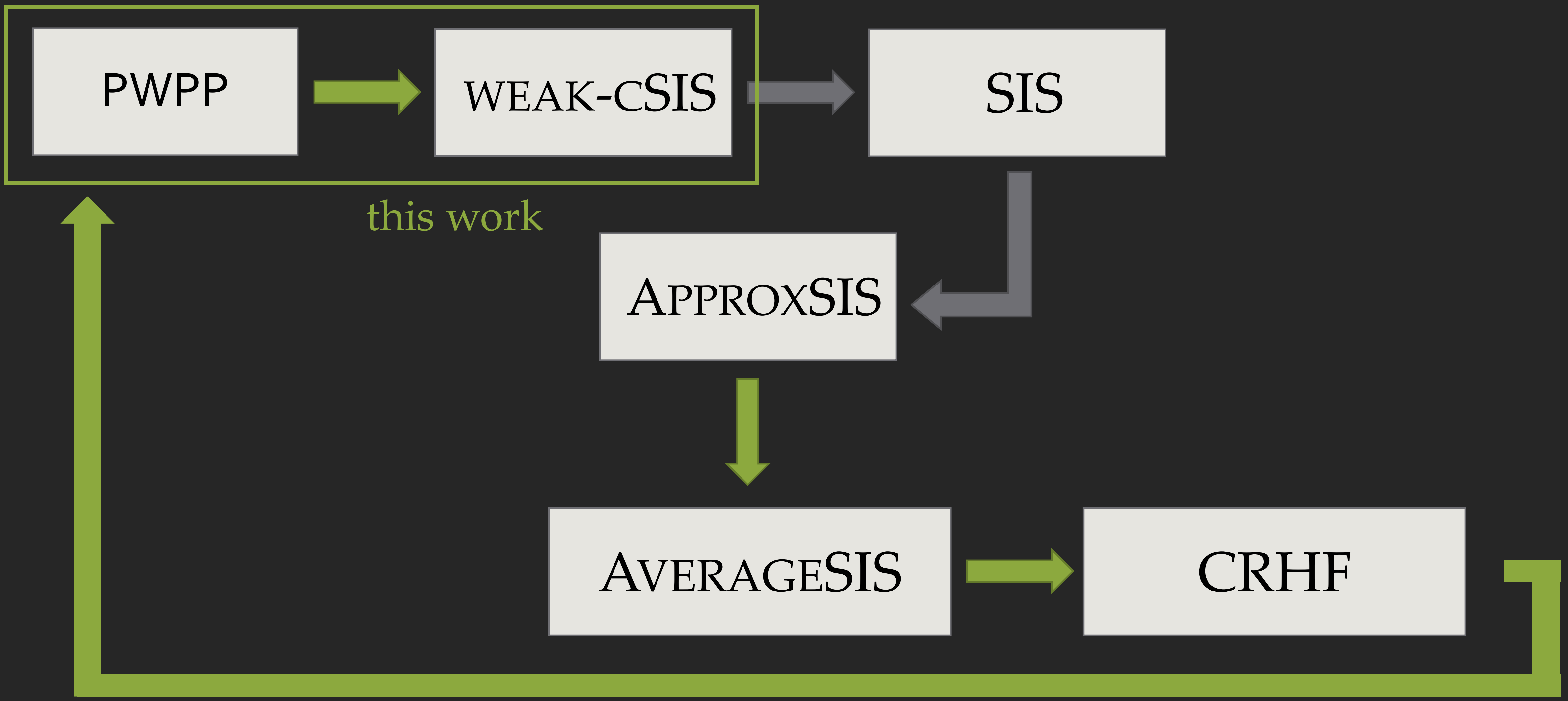

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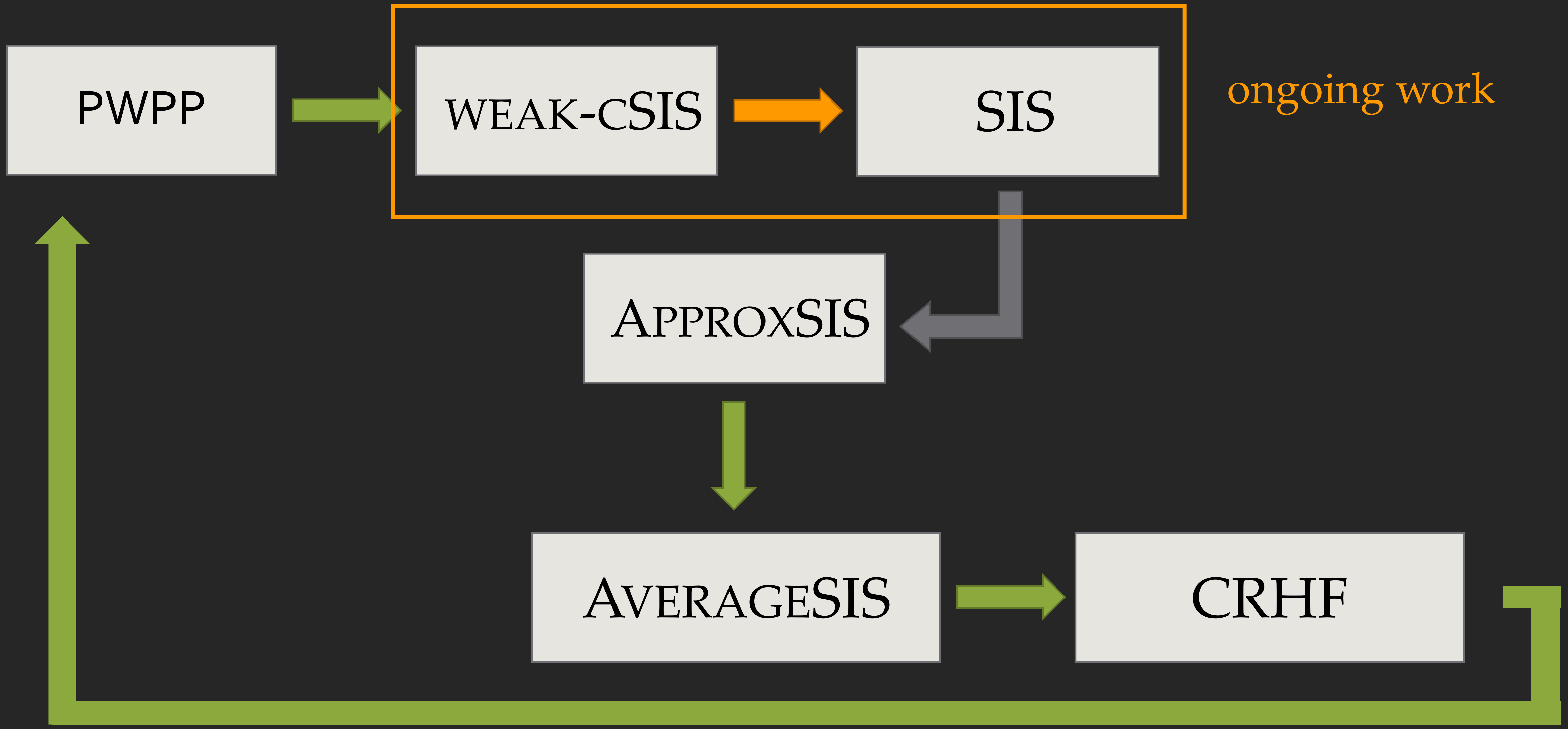

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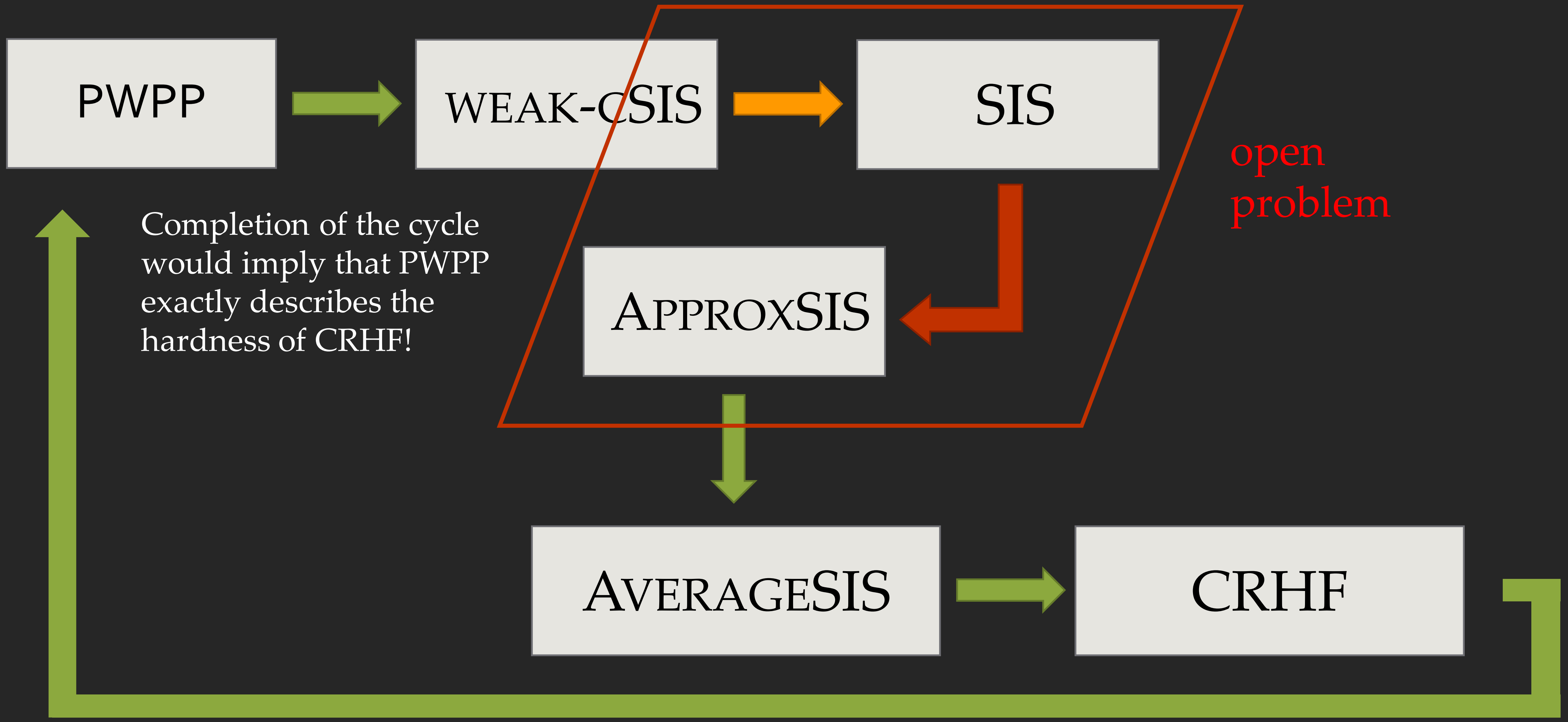
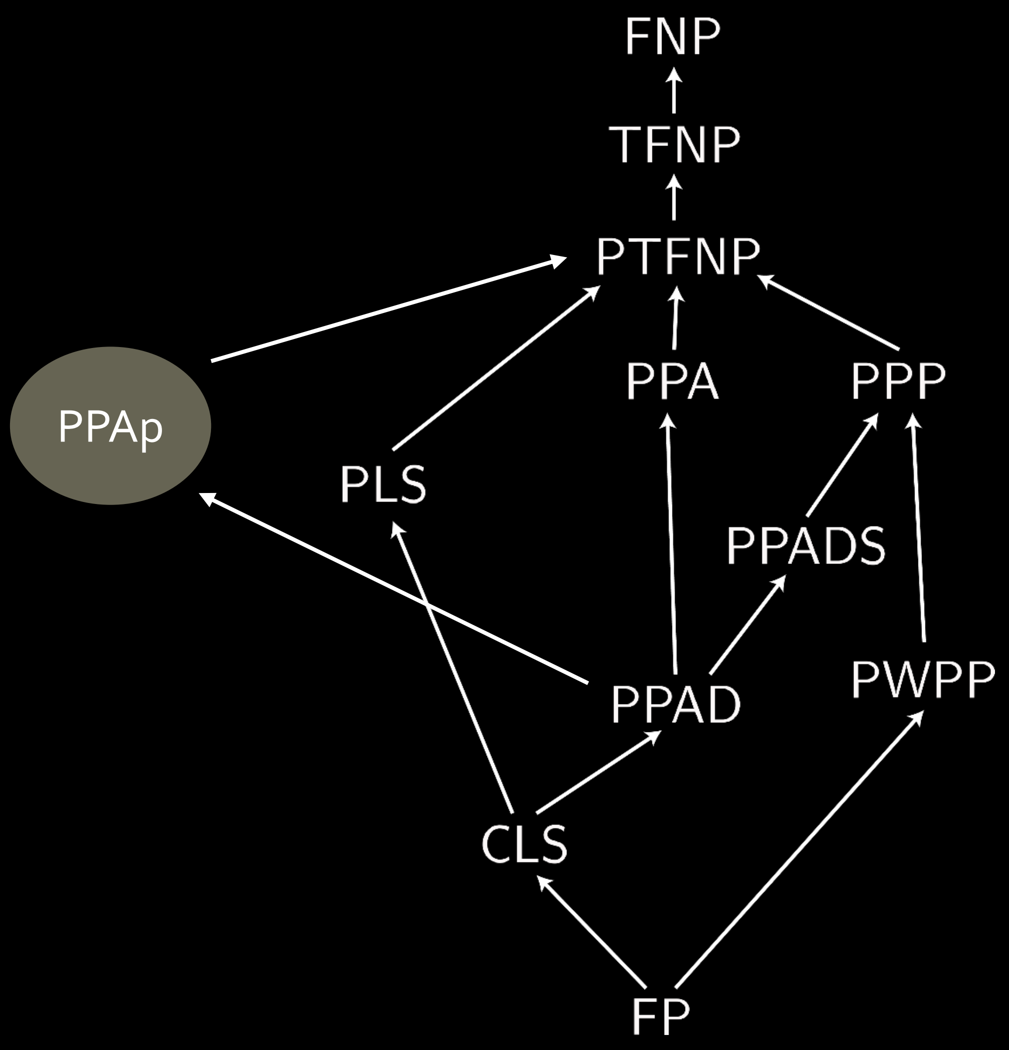


Complexity of Total Search Problems

[Goos Kamath Sotiraki Z.'20]

[Filos-Ratsikas Hollender Sotiraki Z.'20]



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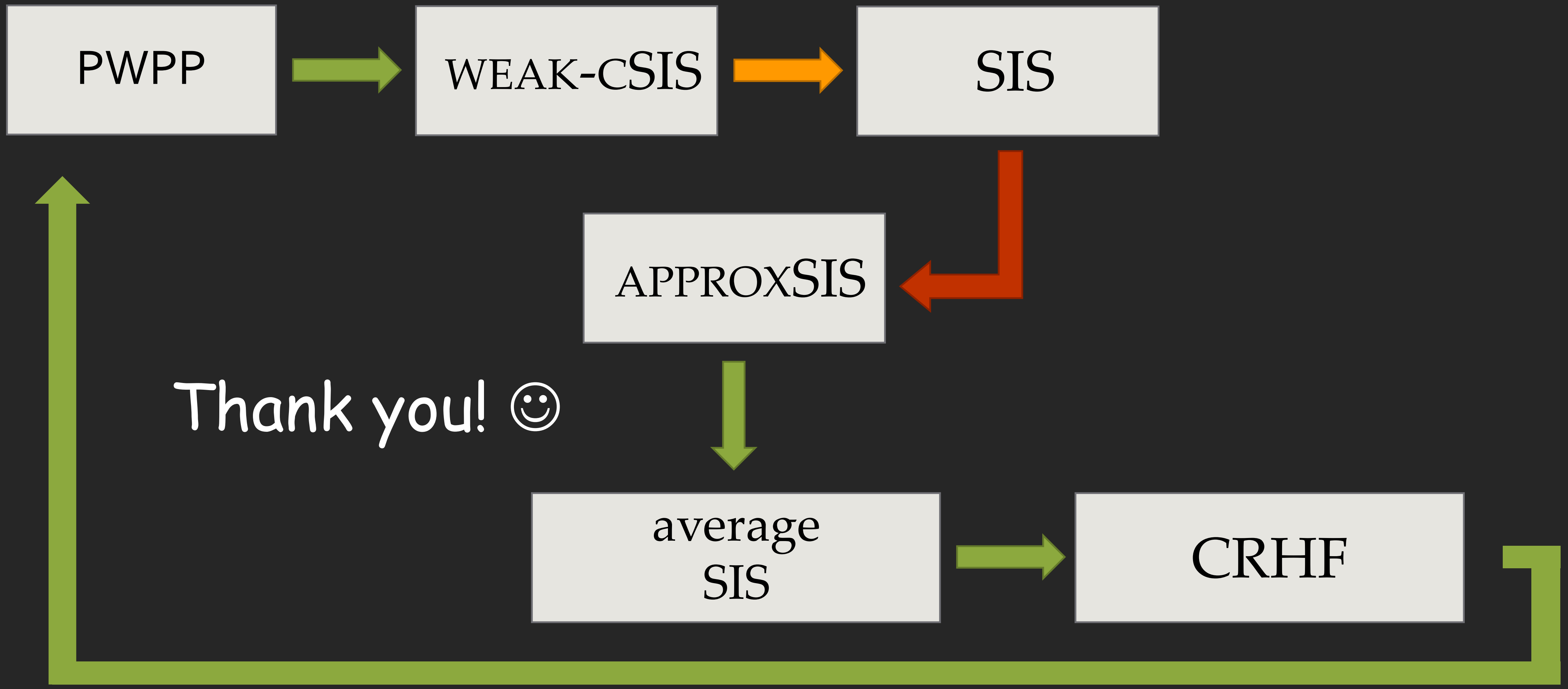
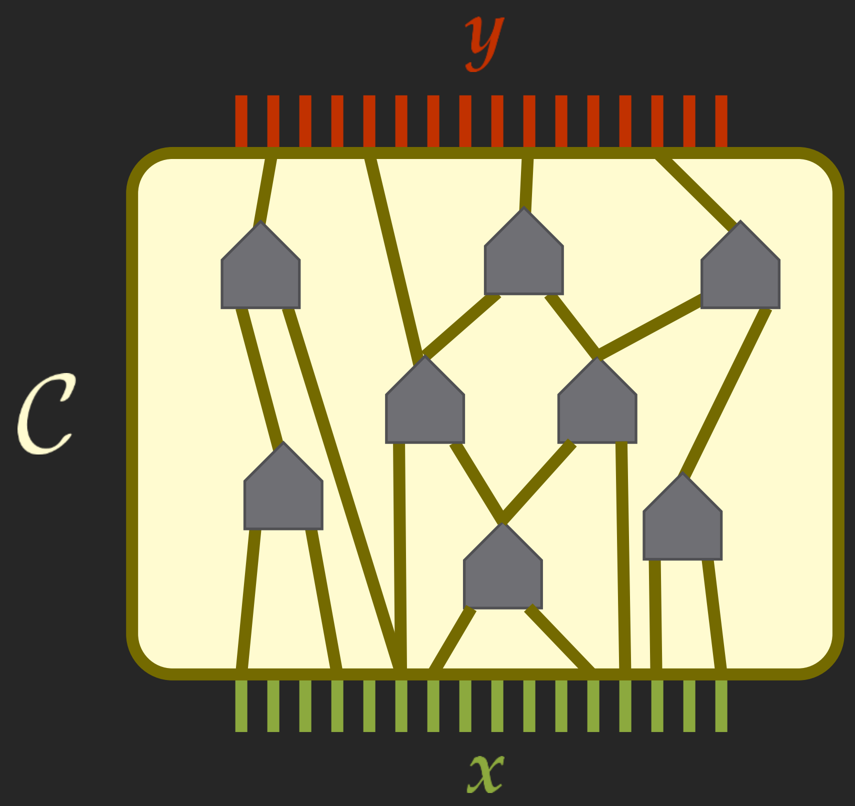


Thank you! ☺

WEAK-cSIS is PWPP-hard



$$\mathbf{G}\begin{bmatrix}\mathbf{y}\\ \\ \mathbf{x}\end{bmatrix} = \mathbf{0} \pmod{q}$$

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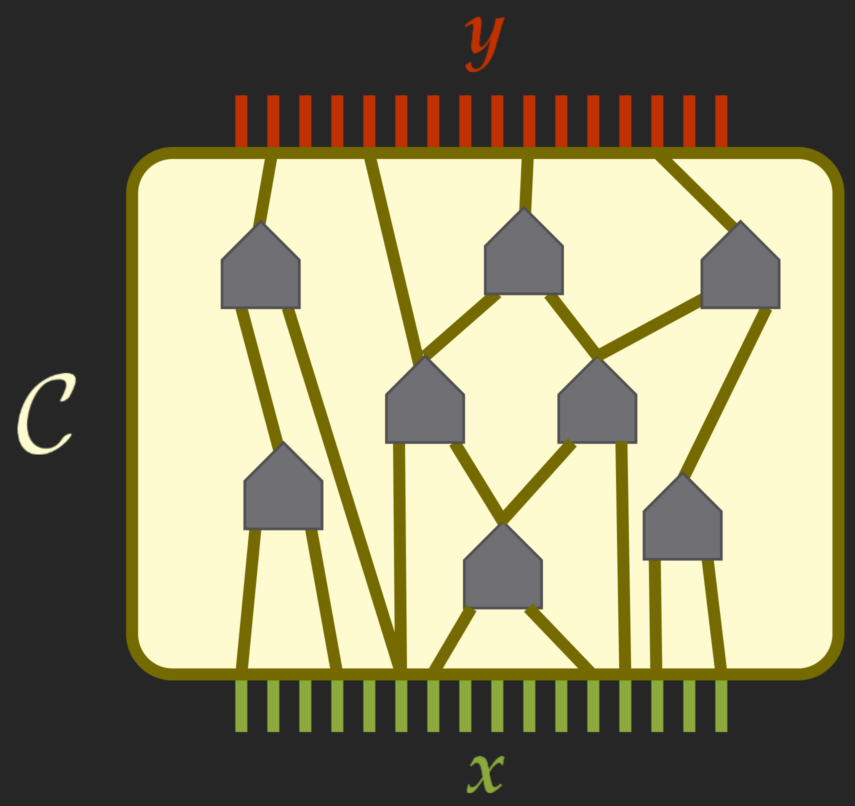
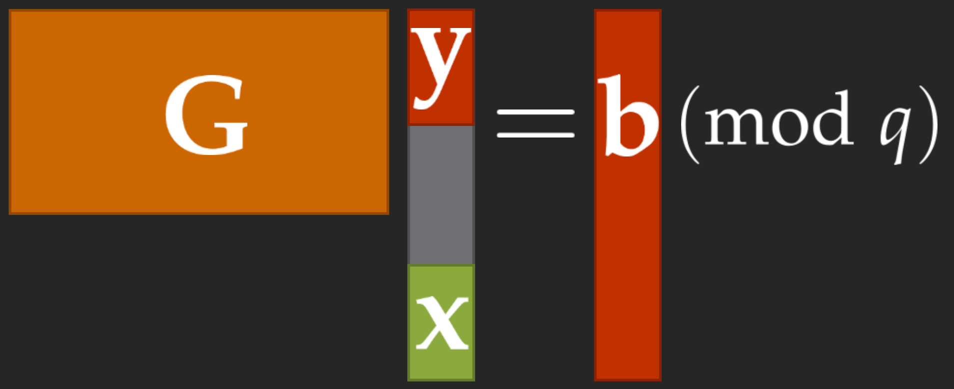
Attention!
During the reduction we have to preserve **totality**!





WEAK-cSIS is PWPP-hard

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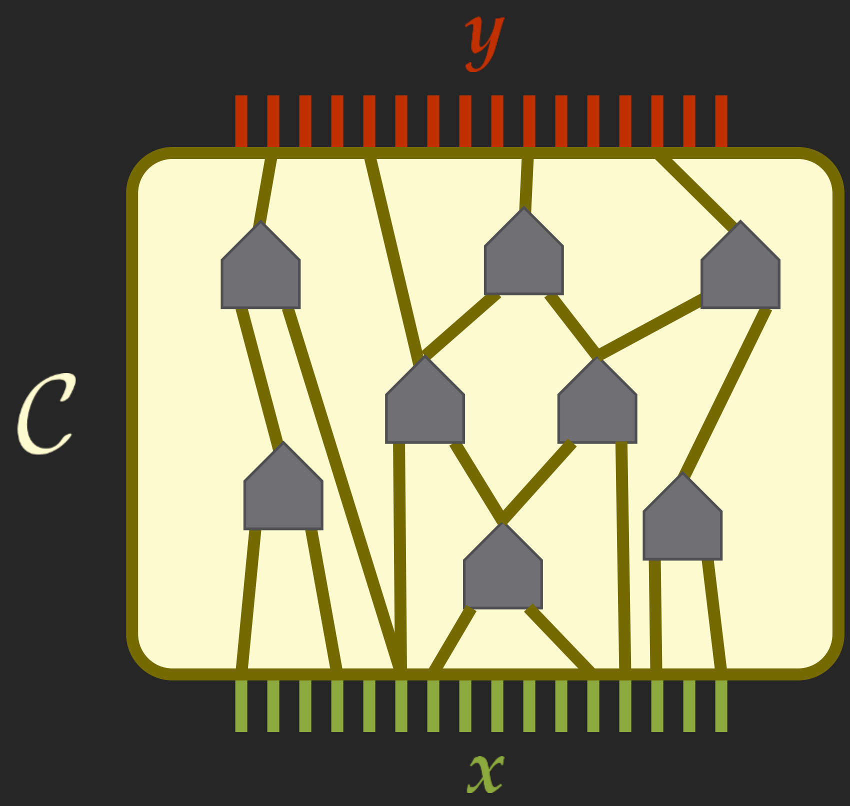
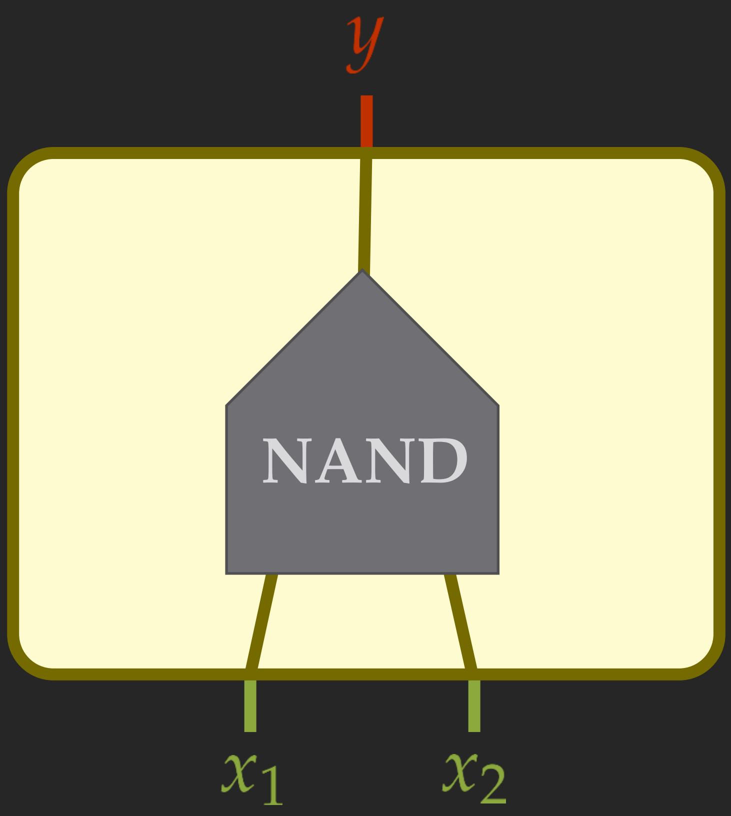


Lemma
For any $\mathbf{z} \in \{0,1\}^{m-\log(q)d}$ and any $\mathbf{b} \in \mathbb{Z}_1^d$, we can efficiently compute a binary solution of the form $\mathbf{x} = [\star \ \ \star \cdots \star \ \ \mathbf{z}]$ for the equation $\mathbf{Gx} = \mathbf{b} \pmod q$.

WEAK-cSIS is PWPP-hard



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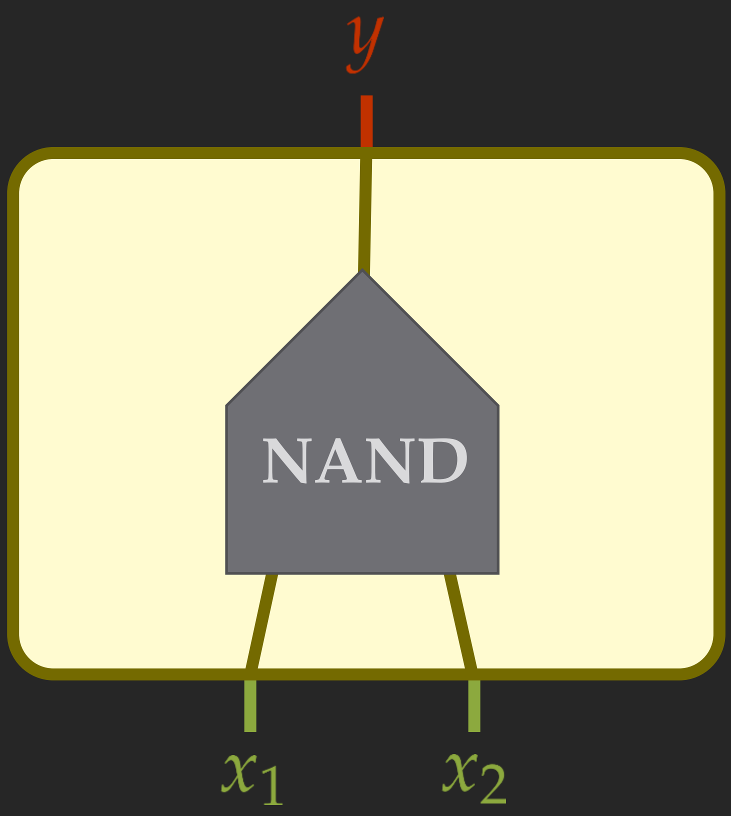
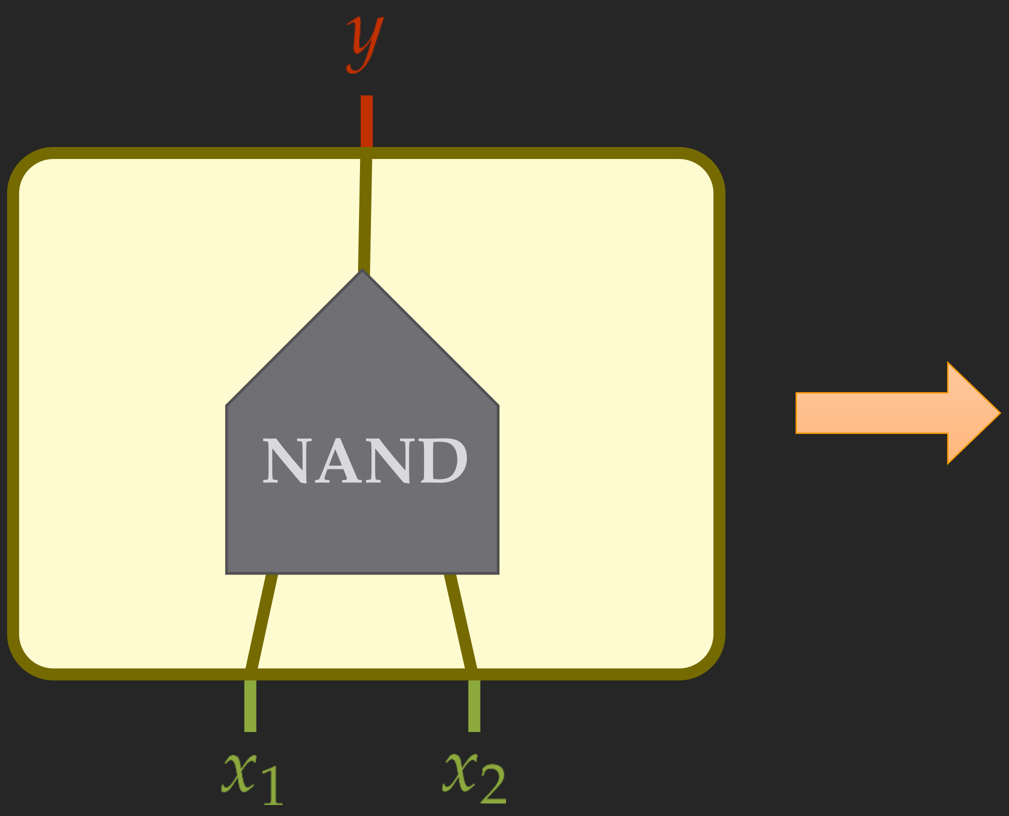


$$1 \cdot v + 2 \cdot y - x_1 - x_2 = 2 \pmod 4$$

WEAK-cSIS is PWPP-hard



$$\underbrace{\boxed{1 \cdot v + 2 \cdot y}}_{\mathbf{g}} - x_1 - x_2 = 2 \pmod 4$$

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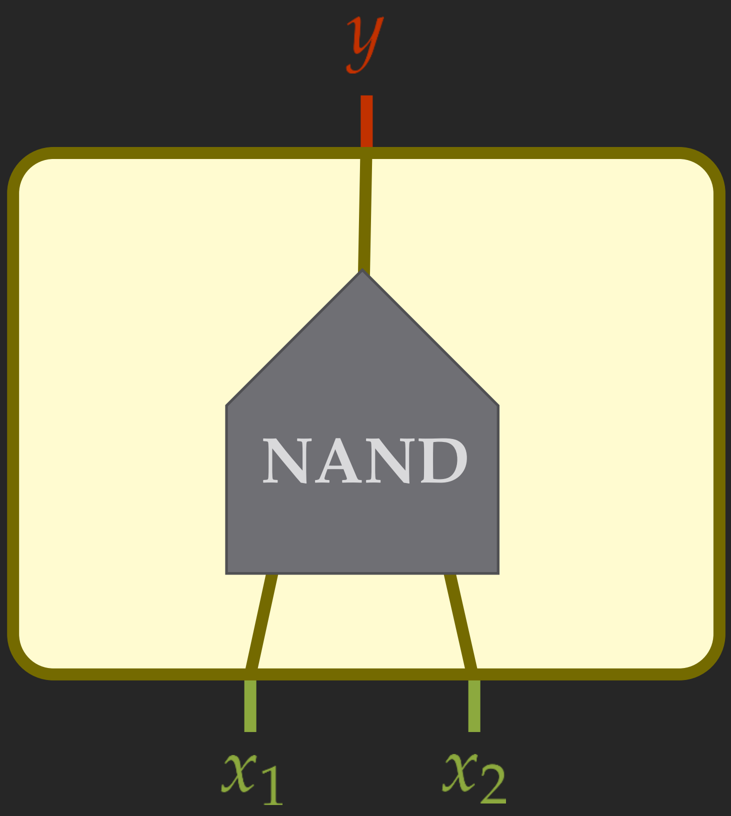
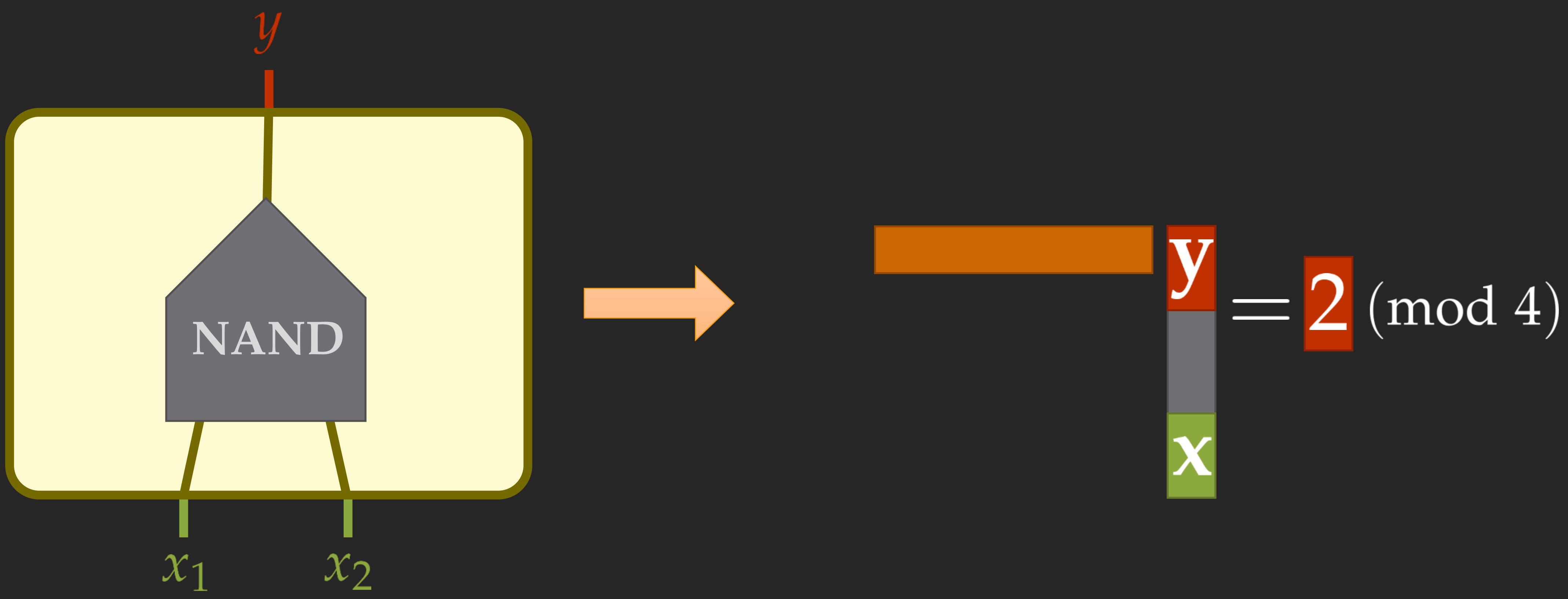


$$1 \cdot v + 2 \cdot y - x_1 - x_2 = 2 \pmod 4$$

0	1	0	0
1	1	0	1
1	1	1	0
0	0	1	1

WEAK-cSIS is PWPP-hard



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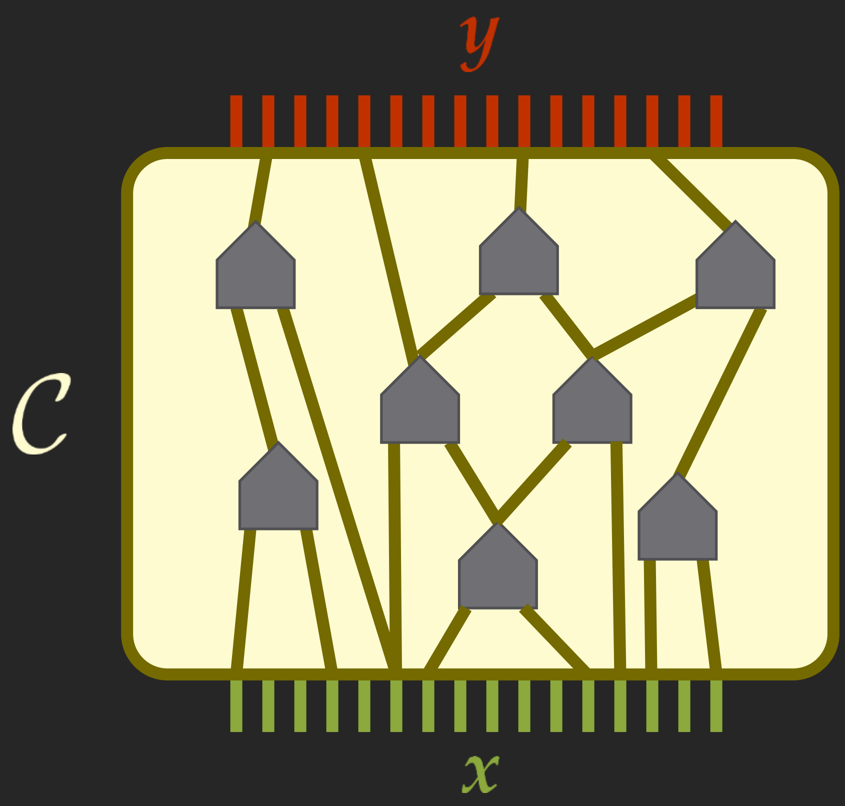

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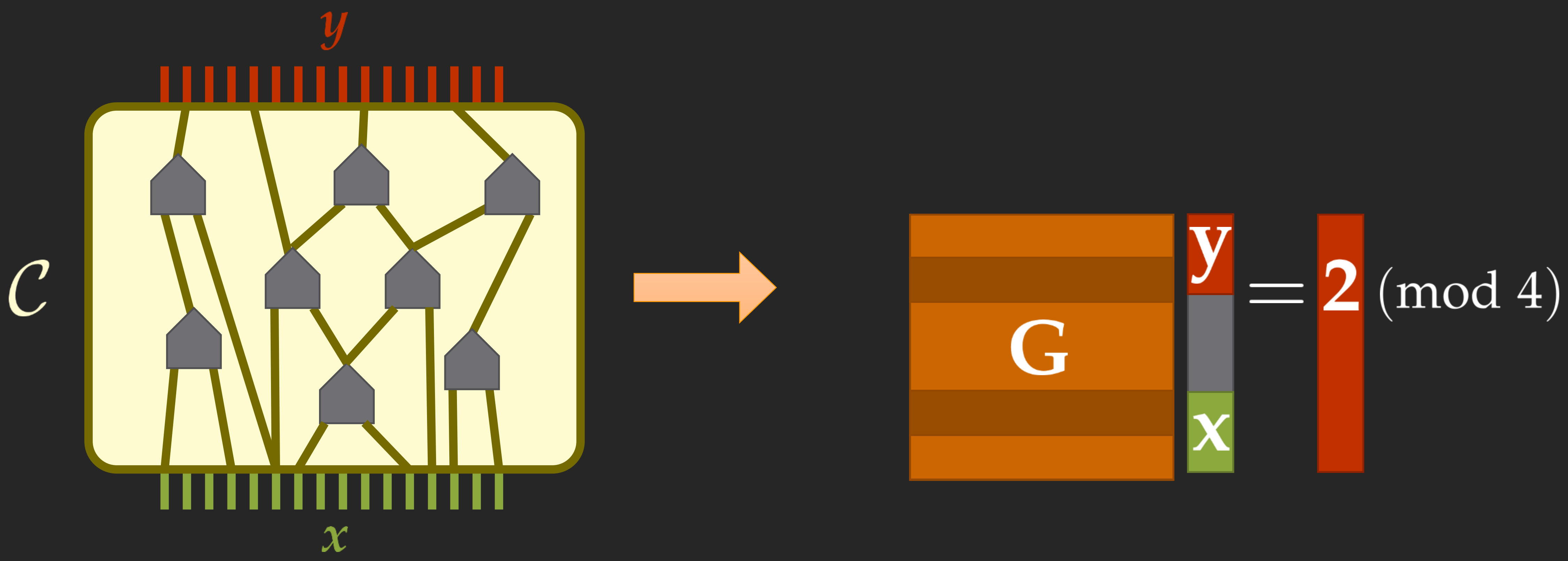

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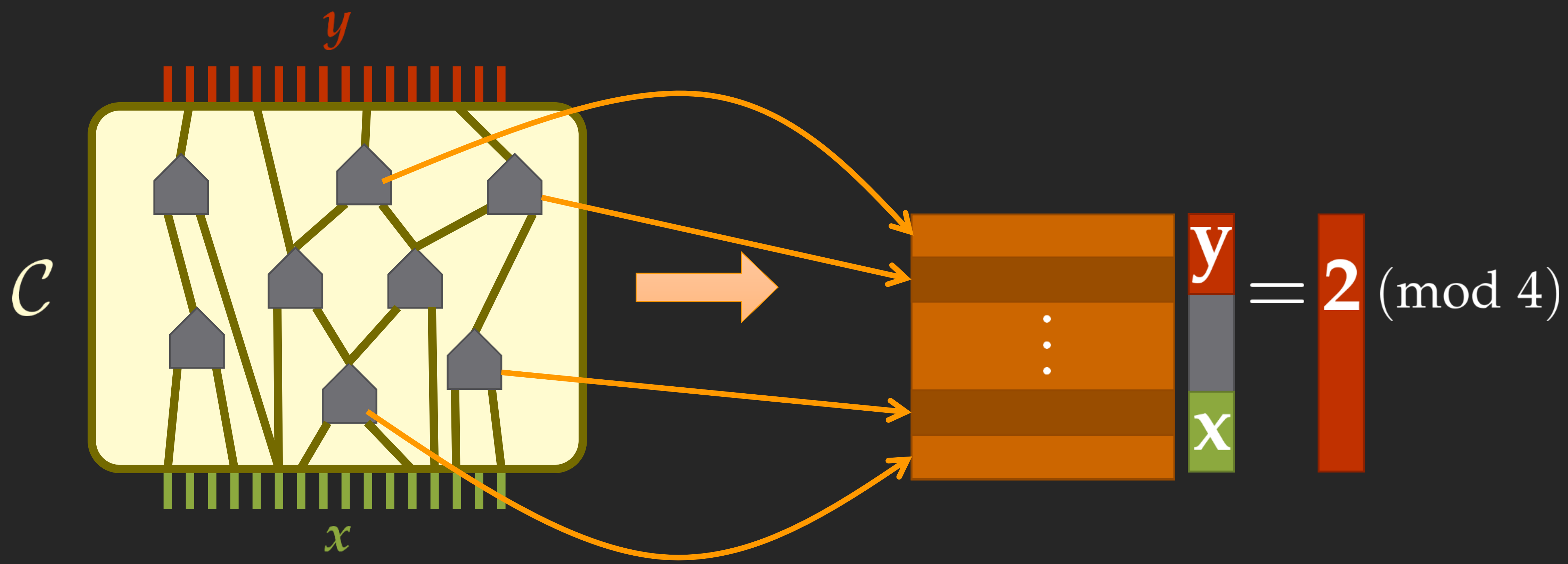

WEAK-cSIS is PWPP-hard

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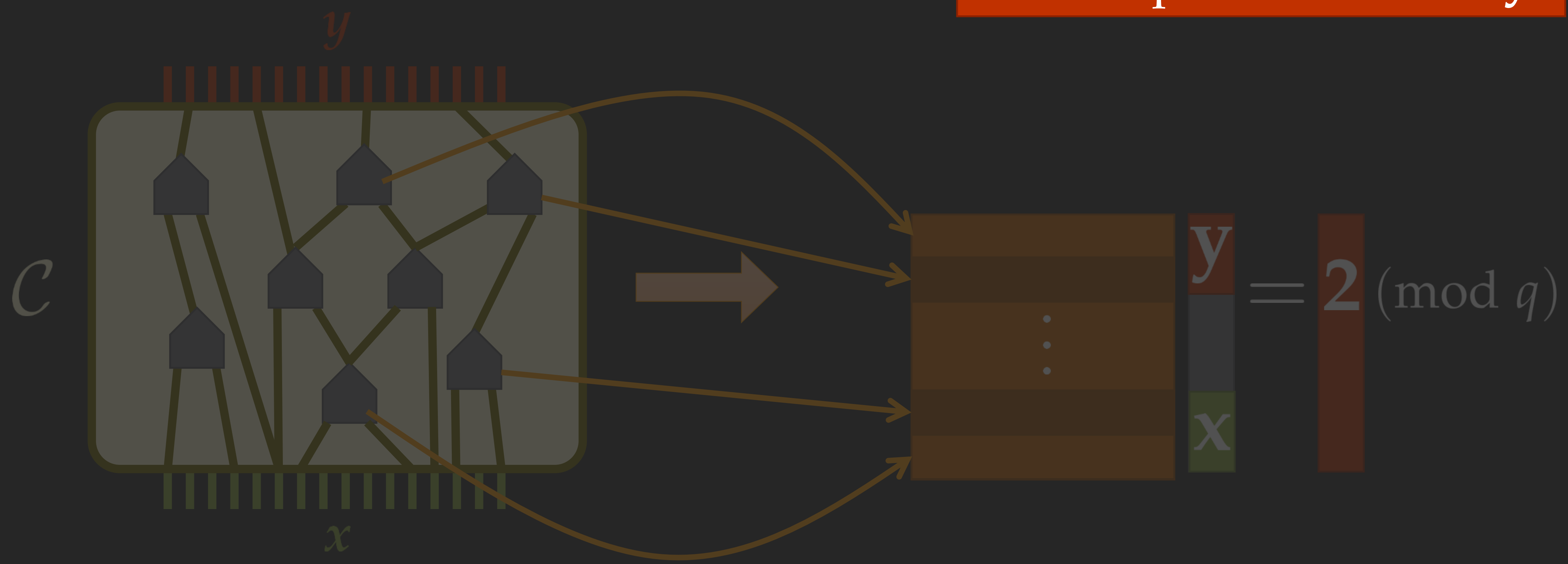
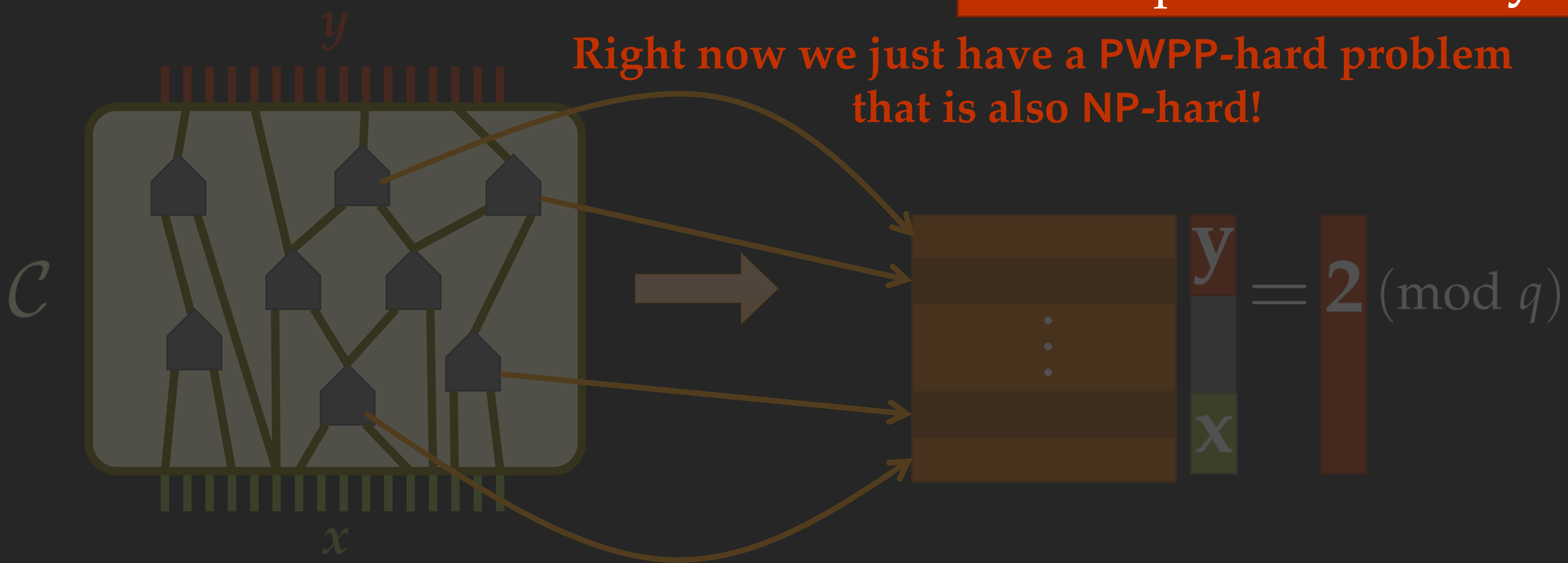


WEAK-CSIS is PWPP-hard

Attention!
During the reduction we have to preserve **totality**!

Right now we just have a PWPP-hard problem that is also NP-hard!



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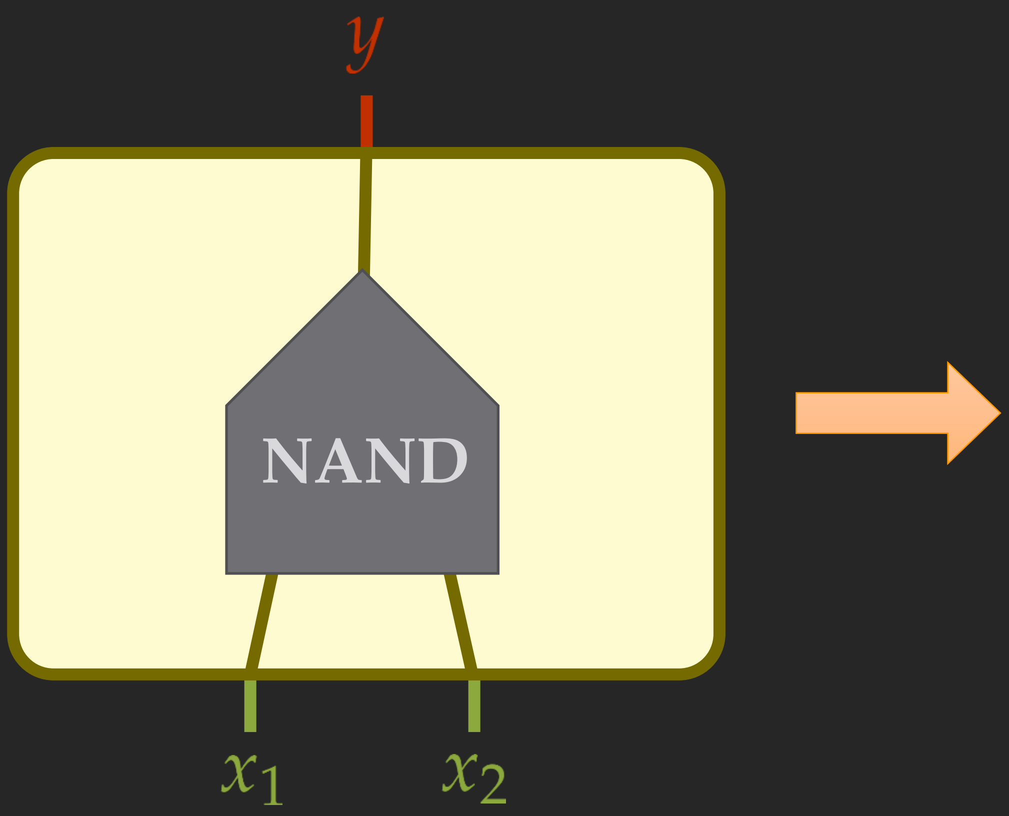
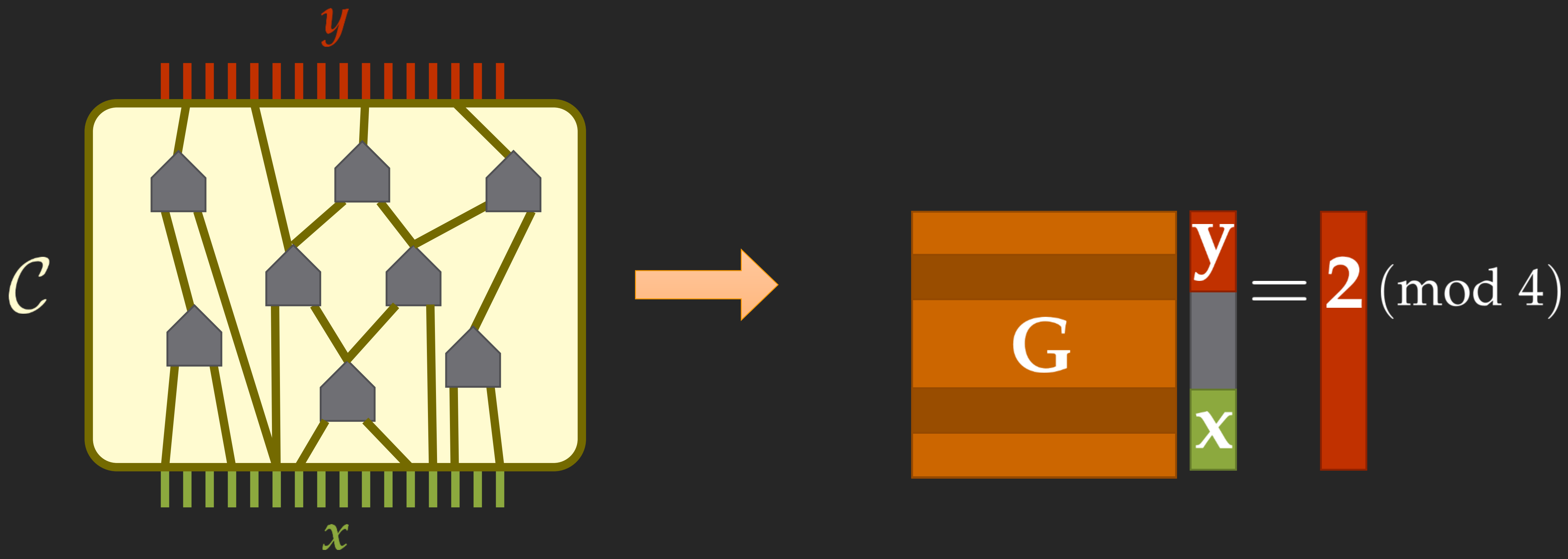


$$\underbrace{\boxed{1 \cdot v + 2 \cdot y}}_{\mathbf{g}} - x_1 - x_2 = 2 \pmod 4$$

WEAK-cSIS is PWPP-hard



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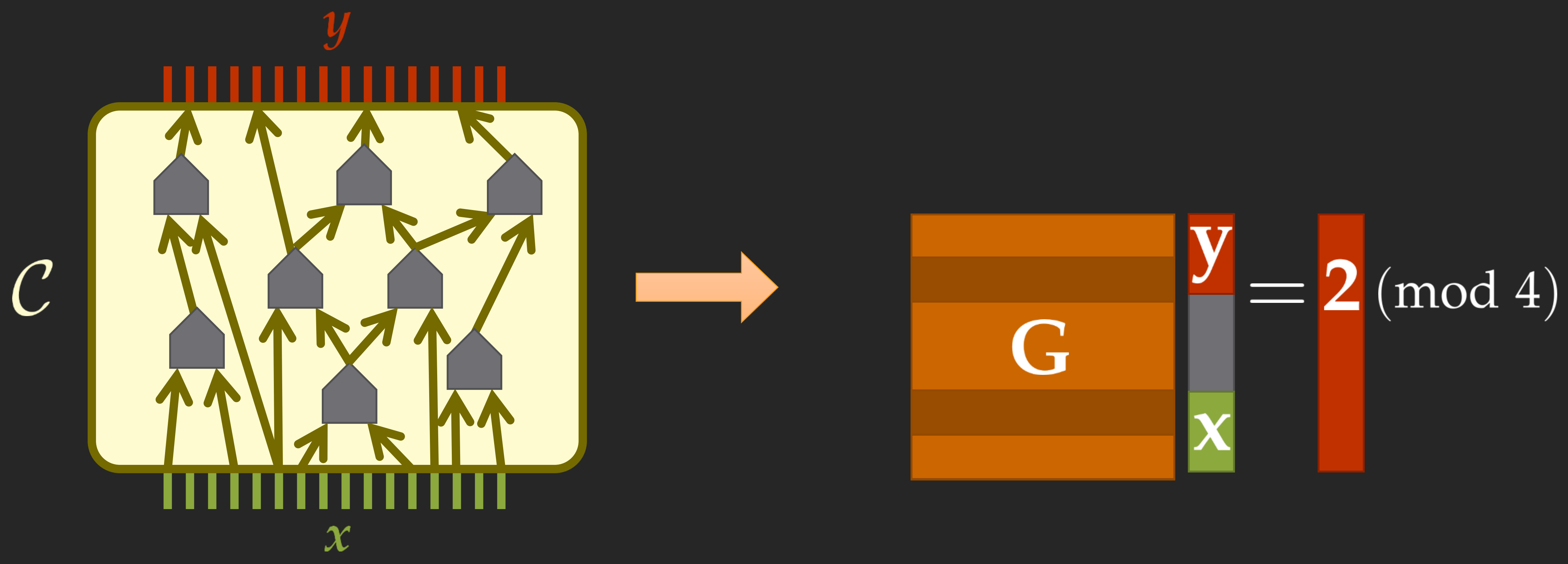

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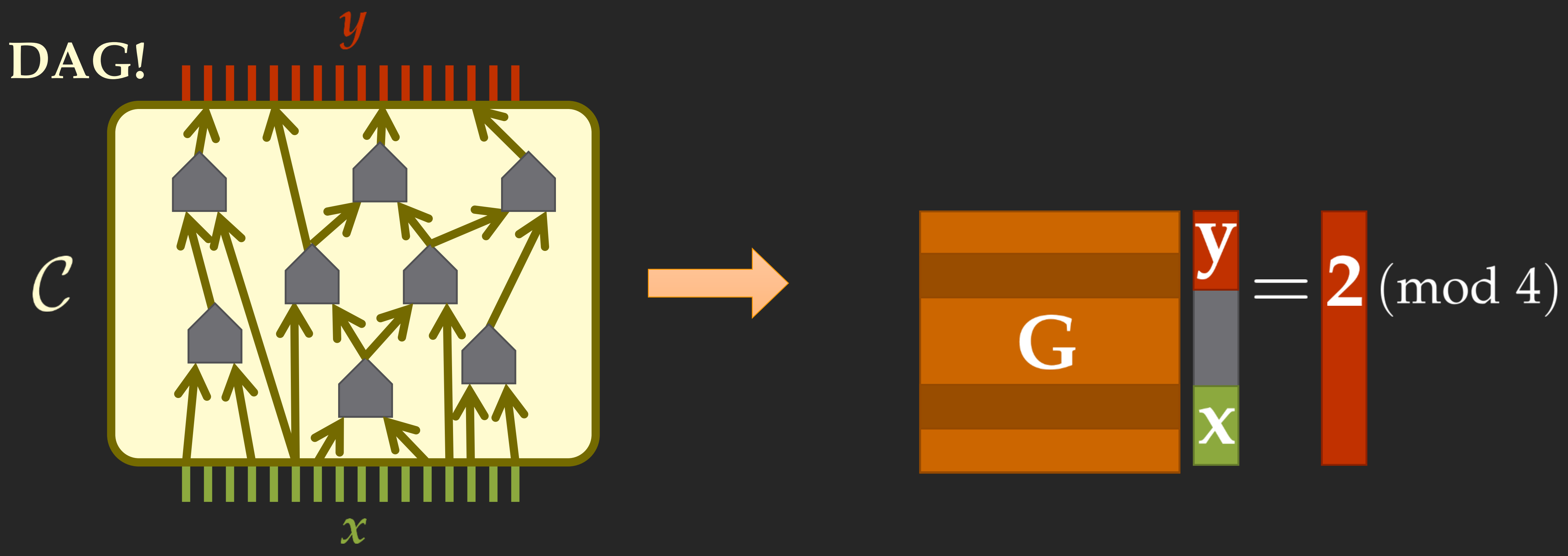

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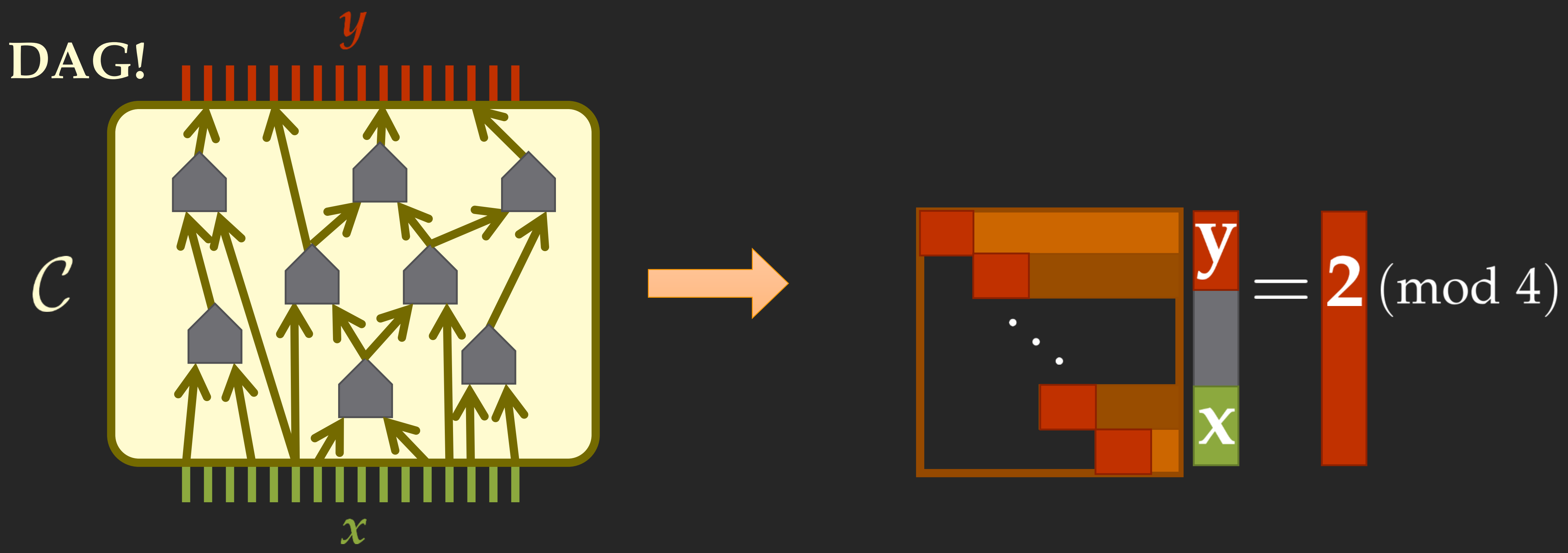

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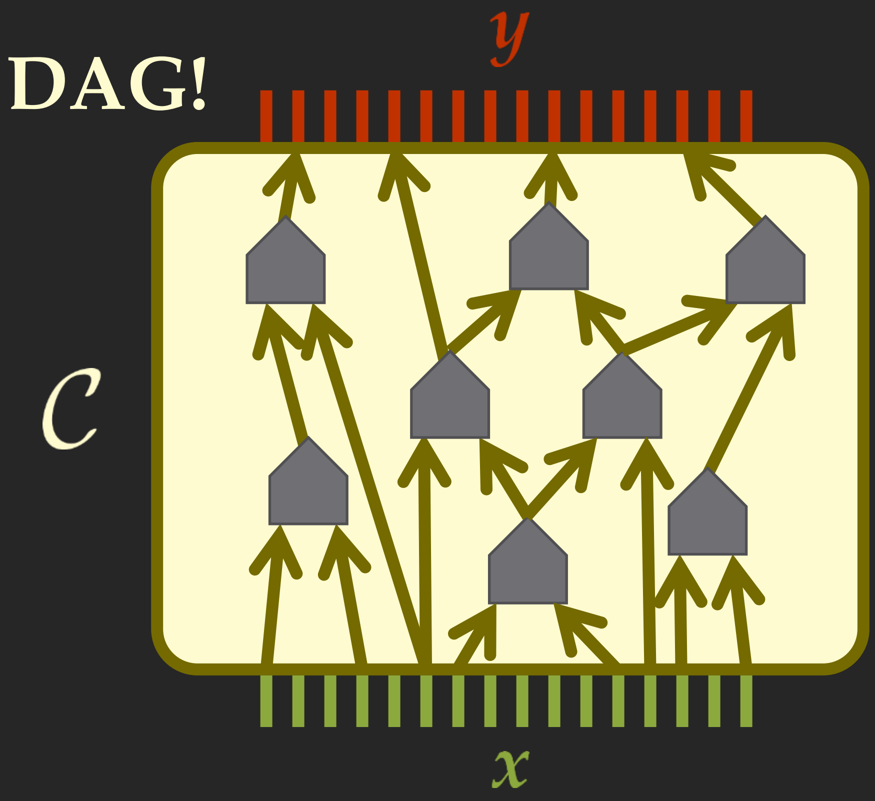
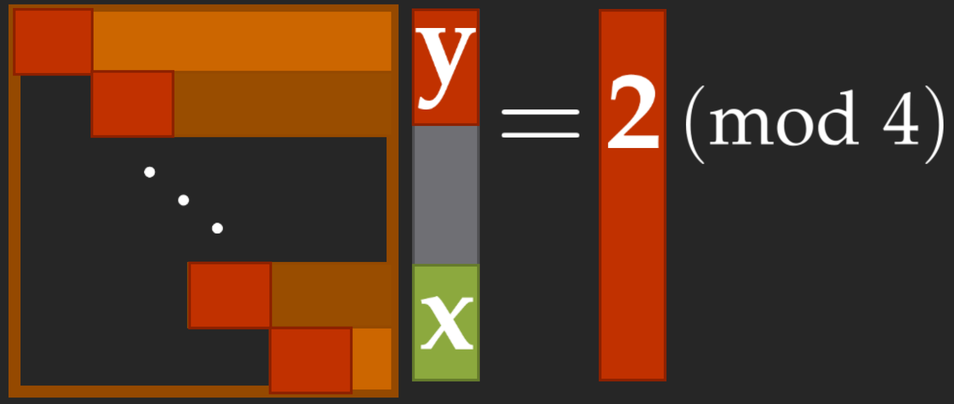
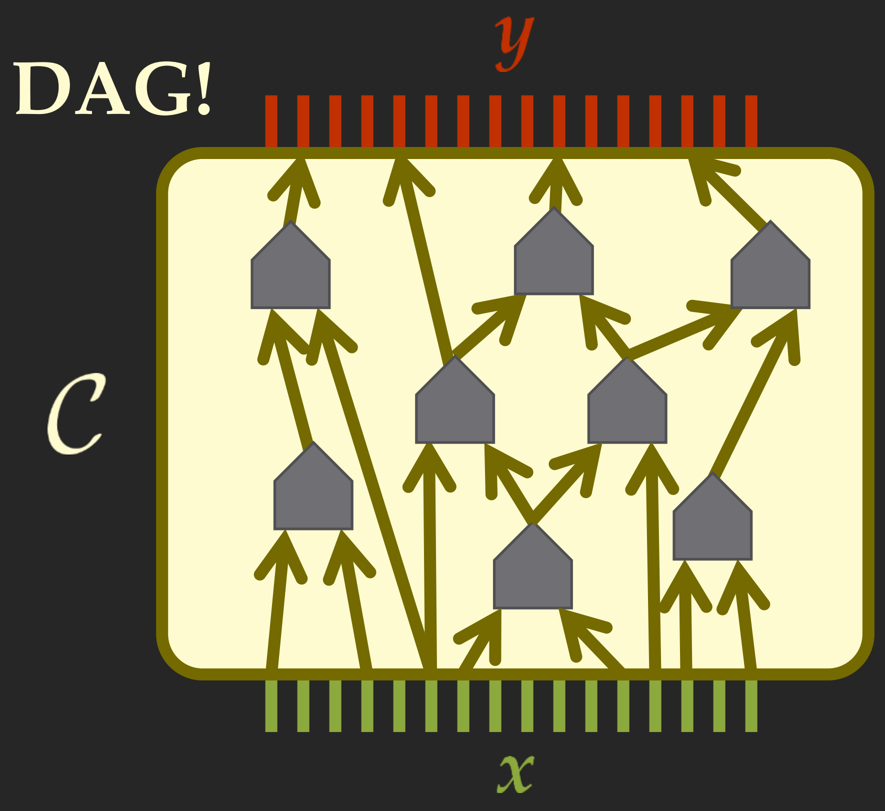
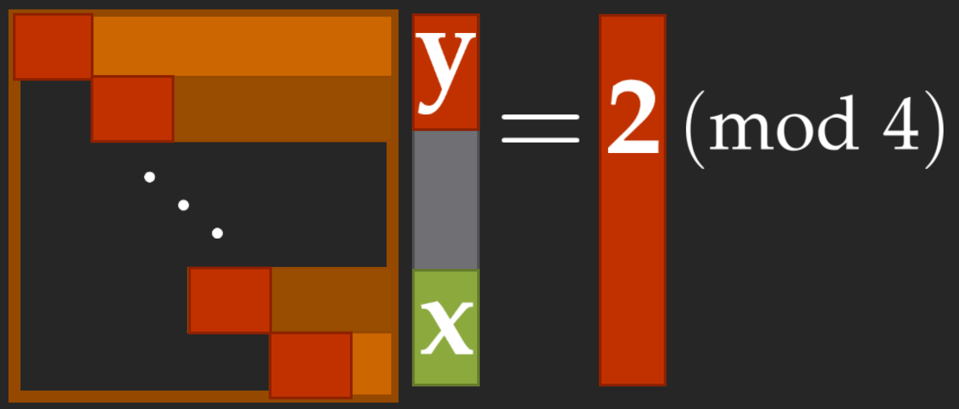




Binary invertible!

WEAK-cSIS is PWPP-hard





Binary invertible!

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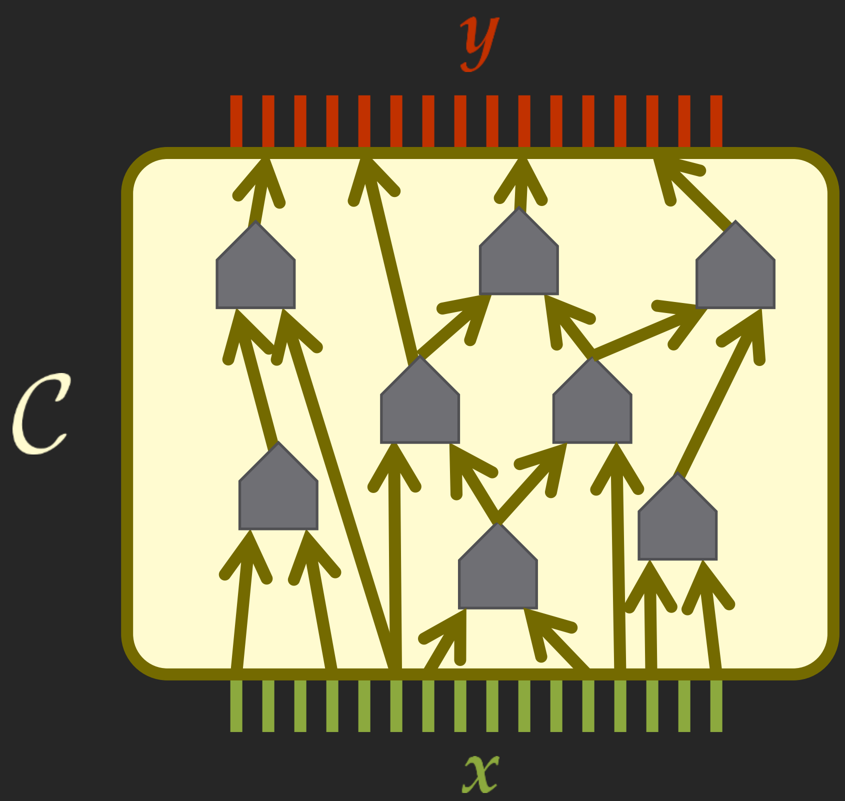

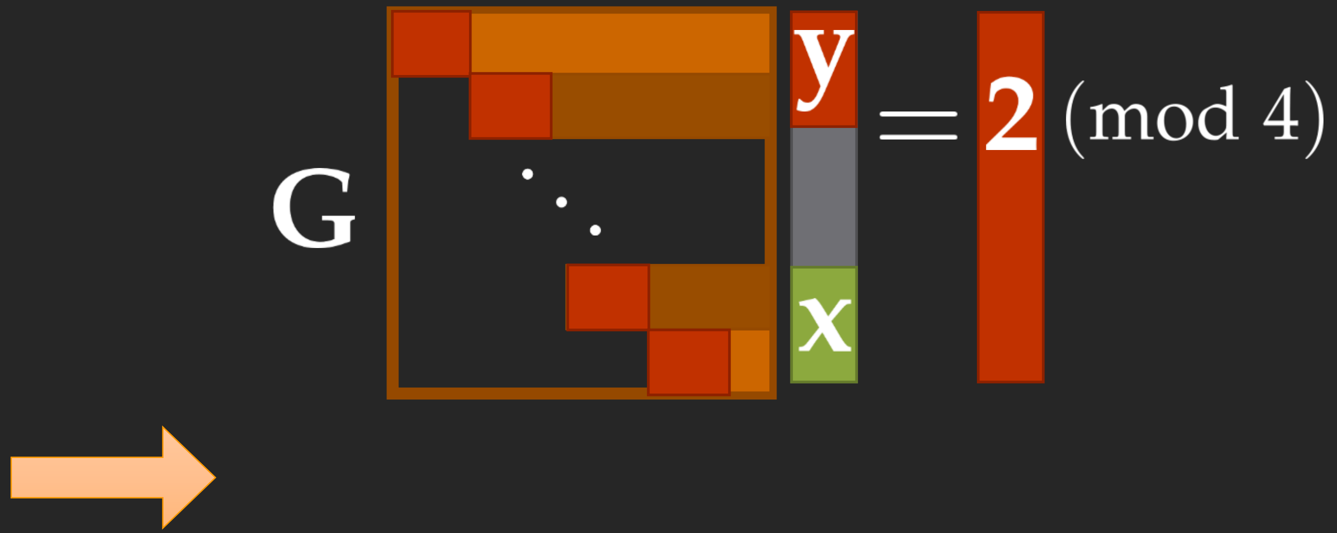

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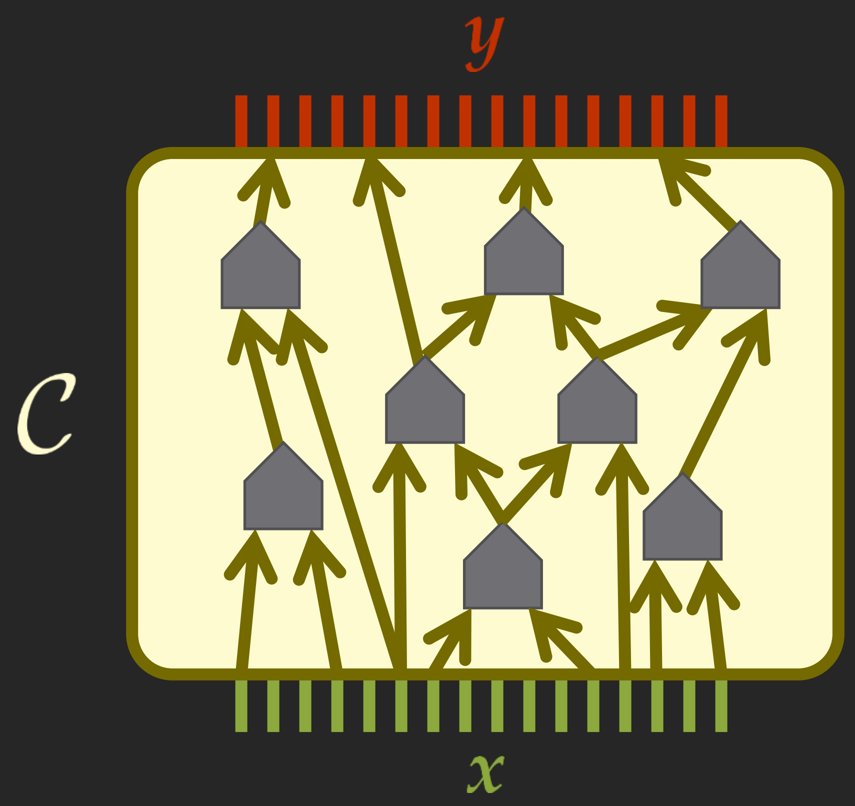

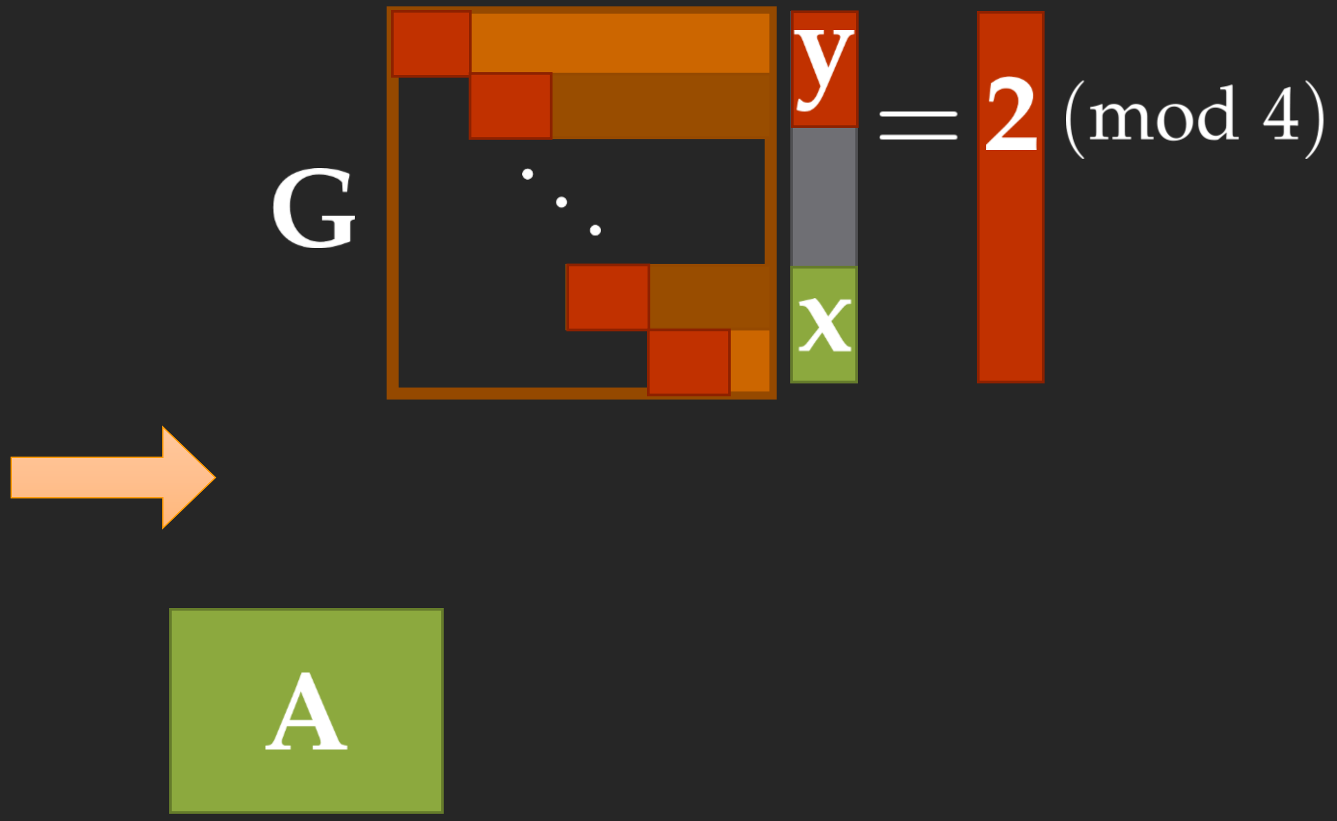

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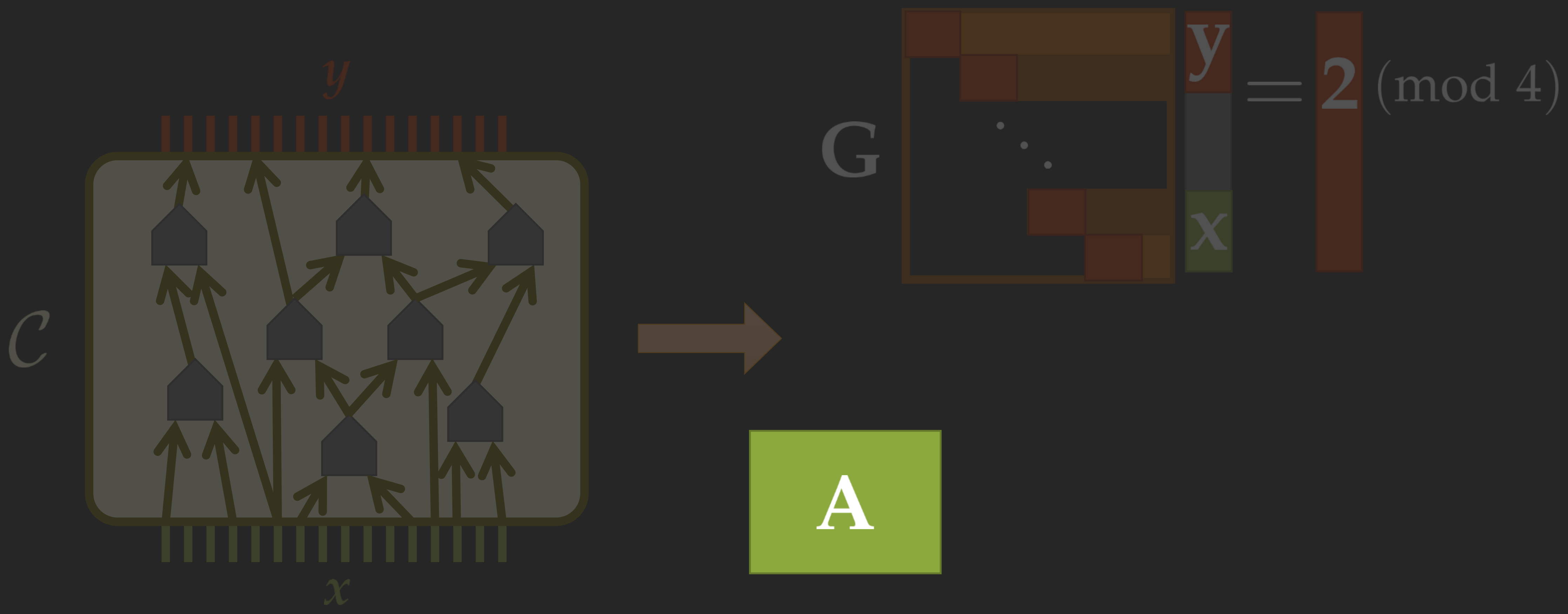

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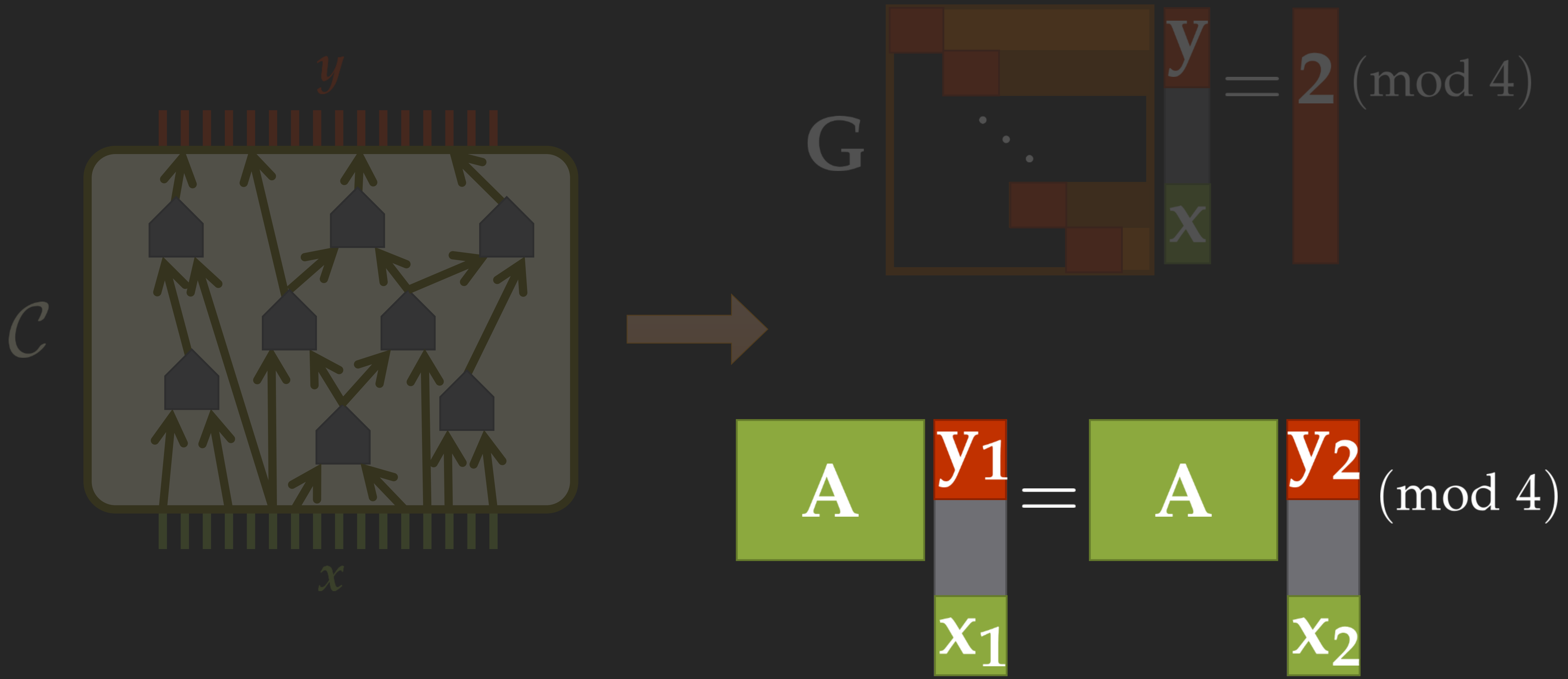

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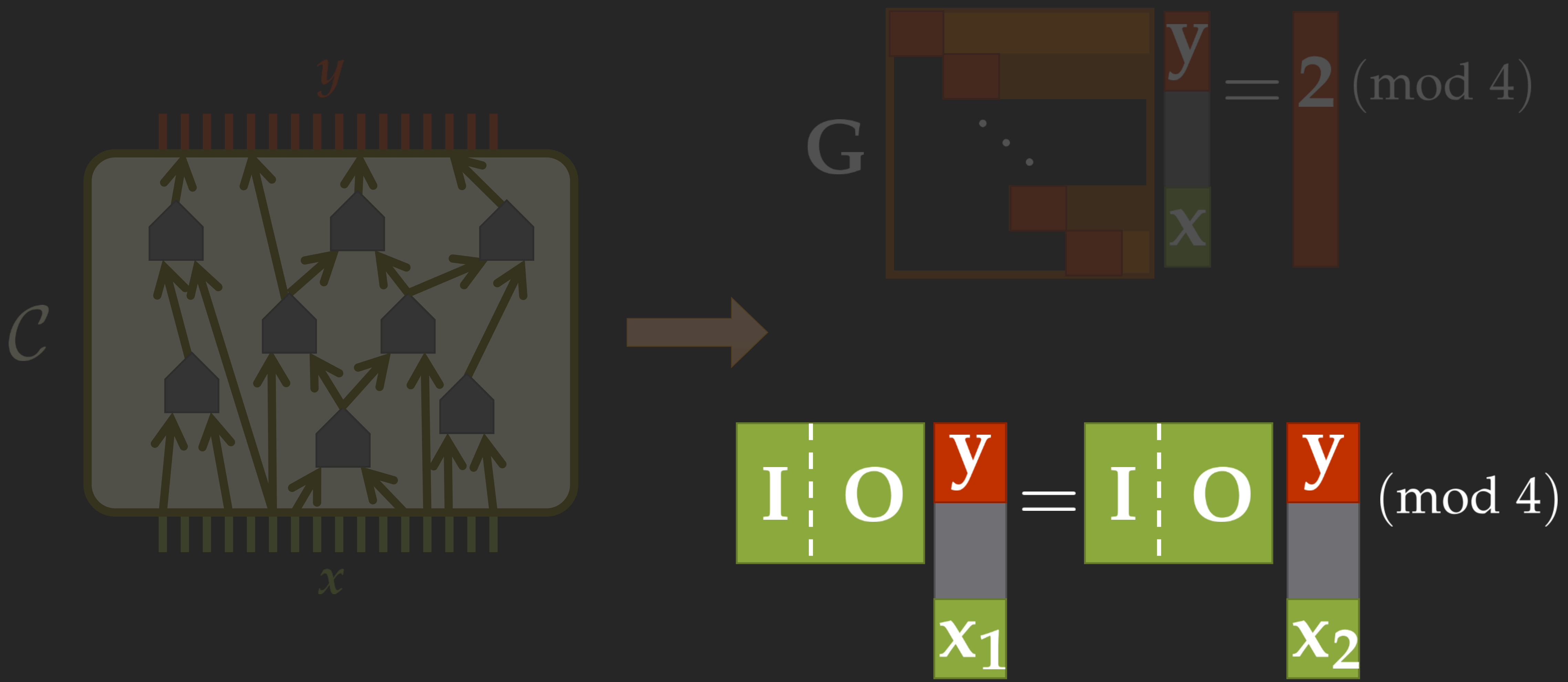

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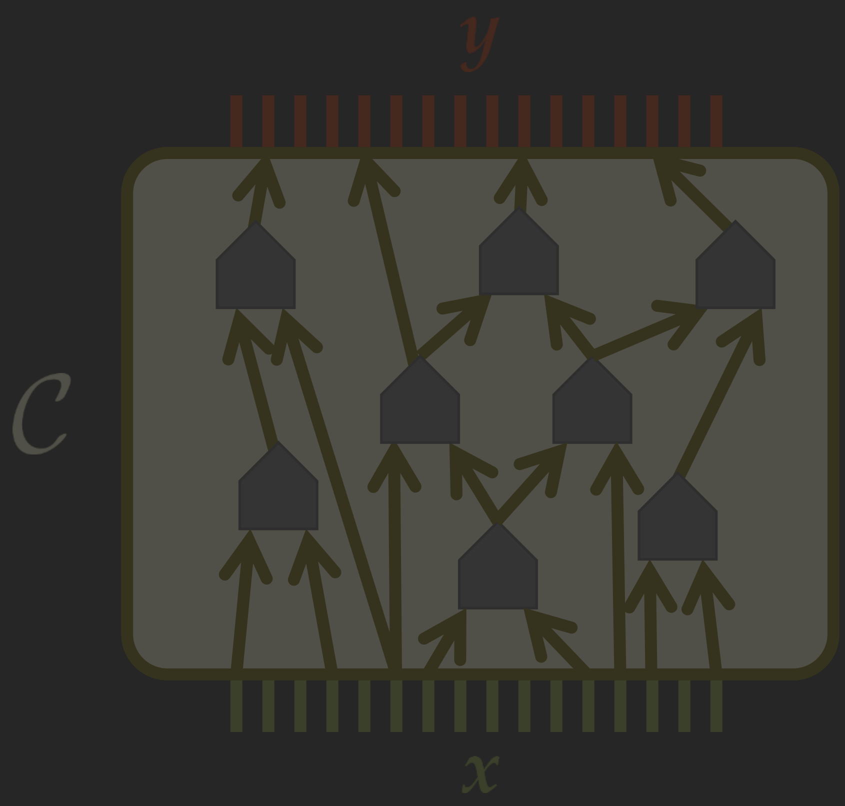
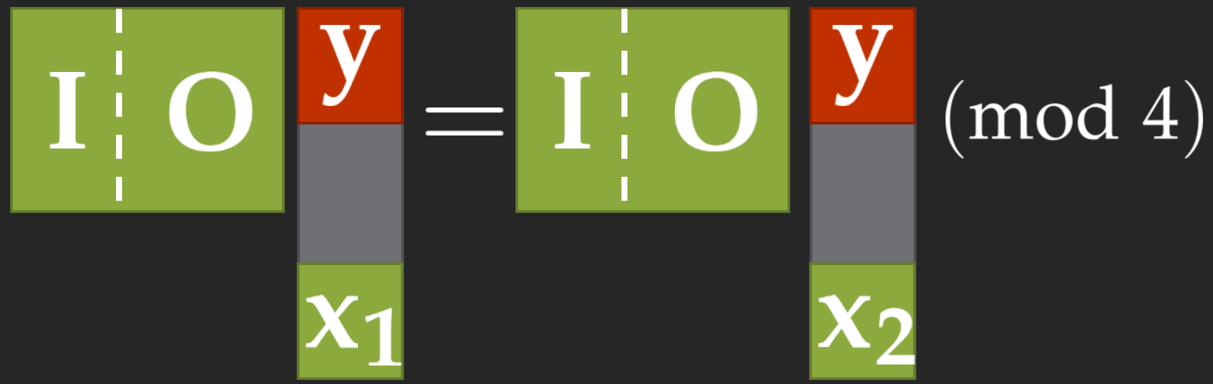




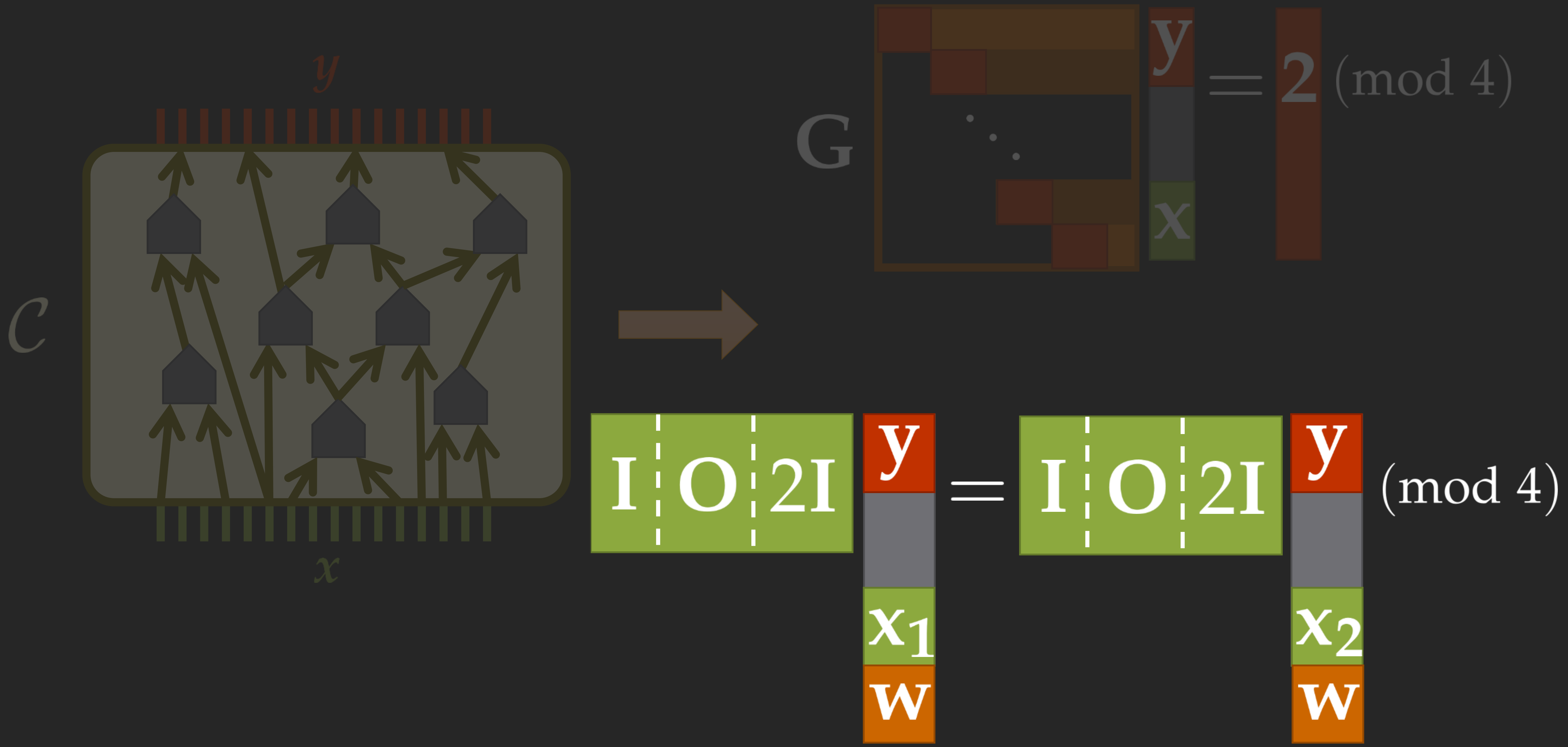
WEAK-CSIS is PWPP-hard

INPUT: $\mathbf{A} \in \mathbb{Z}_q^{r\times m}$, with $m > \log(q)(r+d)$ $\mathbf{G} \in \mathbb{Z}_q^{d\times m}$, and *binary invertible*

OUTPUT: $\mathbf{x}\ \mathbf{y} \in \{0,1\}^m$ s.t. $\mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{y} \pmod q$

$\mathbf{G}\mathbf{x} = \mathbf{G}\mathbf{y} = \mathbf{0} \pmod q$

WEAK-CSIS is PWPP-hard



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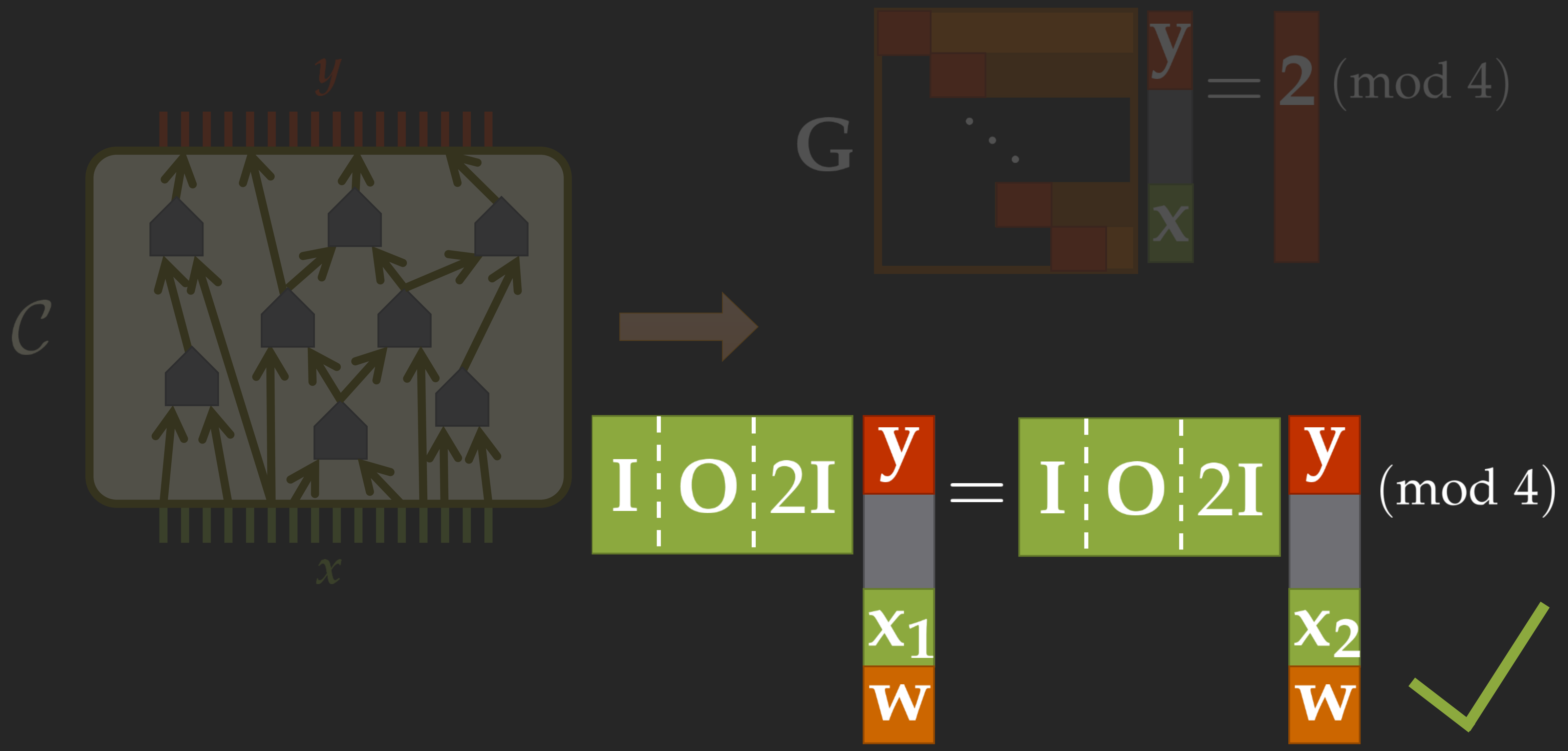

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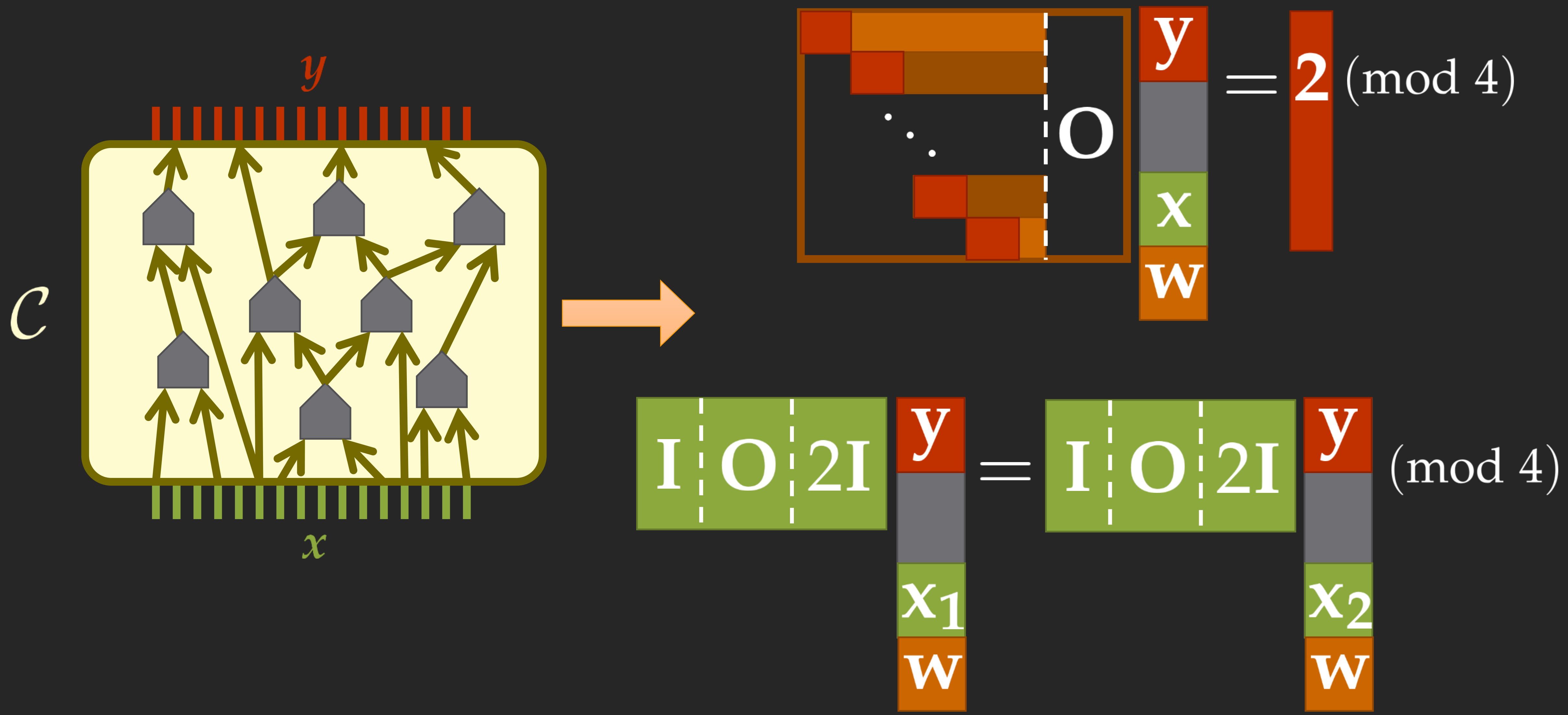

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