Notes in Social Choice

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- Mathematical **theory** dealing with **aggregation** of preferences.
- Founded by Condorcet, Borda (1700's) and Dodgson (1800's).
- Axiomatic framework and impossibility result by Arrow (1951).

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- Collective decision making, by **voting**, over anything:
  - Political representatives, award nominees, contest winners, allocation of tasks/resources, joint plans, meetings, food, ...
  - Web-page ranking, preferences in multiagent systems.

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Colors of the Local Football Club?

Preferences of the founders about the **colors** of the local club:

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A Class of Voting Rules

Positional Scoring Voting Rules

- Vector $(a_1, \ldots, a_m)$, $a_1 \geq \cdots \geq a_m \geq 0$, of **points** allocated to each **position** in the preference list.
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- "Approximation" of the Condorcet winner:
  Dodgson (NP-hard to approximate!), **Copeland**, **MiniMax**, . . .

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Desirable Properties of Social Choice Functions

- Onto : Range is A.
- Unanimous : If a is the top alternative in all $\succ_1, \ldots, \succ_n$, then
$$F(\succ_1, \ldots, \succ_n) = a$$
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- **Strategyproof** or **truthful** : $\forall \succ_1, \ldots, \succ_n$, $\forall$ agent i, $\forall \succ_i'$,
$$F(\succ_1, \ldots, \succ_i, \ldots, \succ_n) \succ_i F(\succ_1, \ldots, \succ_i', \ldots, \succ_n)$$

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Gibbard-Satterthwaite Theorem (mid 70's)

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- Voting systems **computationally hard** to manipulate.

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- Randomization
- Monetary payments
- Voting systems **computationally hard** to manipulate.
- **Restricted domain** of preferences – **Approximation**

Single Peaked Preferences and Medians

Single Peaked Preferences

- One dimensional ordering of alternatives, e.g. $A = [0, 1]$
- Each agent i has a **single peak** $x_i^* \in A$ such that for all $a, b \in A$:

$$b < a \leq x_i^* \quad \Rightarrow \quad a \succ_i b$$
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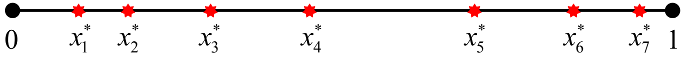

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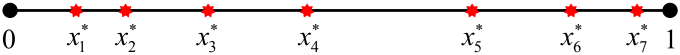
Median Voter Scheme [Moulin 80], [Sprum 91], [Barb Jackson 94]

A social choice function F on a single peaked preference domain is **strategyproof**, **onto**, and **anonymous** iff there exist $y_1, \ldots, y_{n-1} \in A$ such that for all $(x_1^*, \ldots, x_n^*)$,

$$F(x_1^*, \ldots, x_n^*) = \text{median}(x_1^*, \ldots, x_n^*, y_1, \ldots, y_{n-1})$$



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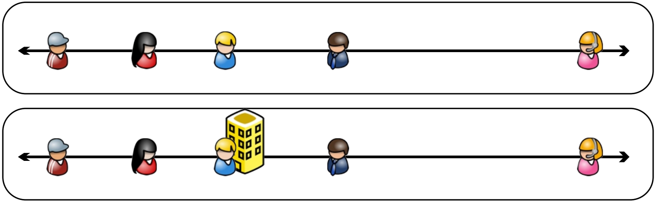
Select a Single Location on the Line

The **median** of $(x_1, \dots, x_n)$ is **strategyproof** (and Condorcet winner) .



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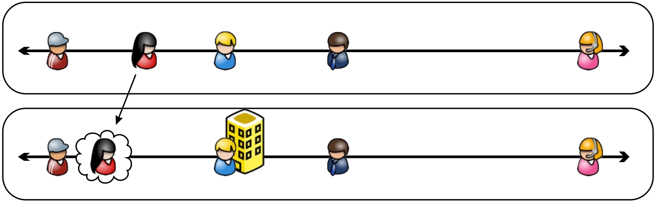

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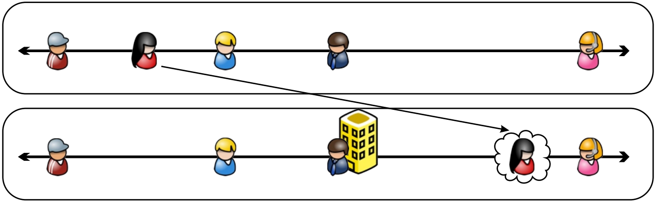

Single Peaked Preferences and Generalized Medians

Generalized Median Voter Scheme [Moulin 80]

A social choice function F on single peaked preference domain $[0, 1]$ is **strategyproof** and **onto** iff it is a **generalized median voter scheme** (GMVS), i.e., there exist 2^n thresholds $\{\alpha_S\}_{S \subset N}$ in $[0, 1]$ such that:

- $\alpha_\emptyset = 0$ and $\alpha_N = 1$ (onto condition),
- $S \subseteq T \subseteq N$ implies $\alpha_S \leq \alpha_T$, and
- for all $(x_1^*, \ldots, x_n^*)$, $F(x_1^*, \ldots, x_n^*) = \max_{S \subset N} \min\{\alpha_S,\ x_i^* : i \in S\}$